THE MATHEMATICS STUDENT

Volume X] JUNE 1942 [Number 2

RULED SURFACES WHOSE CURVED ASYMPTOTIC LINES CAN BE DETERMINED BY QUADRATURES

BY

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Abstract. 1. Picard[1] has shown that if the generators of a ruled surface belong to a linear complex, then the curved asymptotic lines can be determined by quadratures. The determination of the asymptotic lines of some other special kinds of ruled surfaces by quadratures has also been considered by Buhl,[2] Goursat,[3] Srinivasiengar[4] and Hayashi.[5] The object of this paper is to approach the problem of finding ruled surfaces whose curved asymptotic lines can be determined by quadratures in a different way by examining certain conditions under which the Riccati's equation can be integrated.

2. Let the equations of the ruled surface be

$$x = p + lu, \quad y = q + mu, \quad z = r + nu,$$

where $p, q, r; l, m, n$ are functions of v, the arc of the base curve.

The fundamental magnitudes of the second order L, M, N are given by $LV = [x_1, x_2, x_{11}] = [l, p' + ul', 0] = 0$, where the notation $[x_1, x_2, x_{11}]$ denotes

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_{11} & y_{11} & z_{11} \end{vmatrix}$$

$$MV = [x_1, x_2, x_{12}] = [l, p' + ul', l'] = \sum l\ (q'n' - r'm') \equiv \delta, \text{ say},$$

$$NV = [x_1, x_2, x_{22}] = [l, p' + ul', p'' + ul'']$$

$$= \sum l\ (q'\ r'' - r'\ q'') + u \sum l\ \{(m'\ r'' - m''\ r') + (q'\ n'' - n'\ q'')\}$$

$$+ u^2 \sum l\ (m'\ n'' - n'\ m'') \equiv \lambda + \mu u + \nu u^2, \text{ say},$$

where $\lambda = \sum l\ (q'\ r'' - r'\ q'')$,

$$\mu = \sum l\ \{(m'\ r'' - m''\ r') + (q'\ n'' - n'\ q'')\},$$

$$\nu = \sum l\ (m'\ n'' - n'\ m'').$$

The general equation of the asymptotic lines on the surface, $Ldu^2+2Mdudv+Ndv^2=0$, reduces to $dv(2Mdu+Ndv)=0$. Hence one system of asymptotic lines is given by $v=$const., i.e. by the generators. The second system of asymptotic lines viz. the system of curved asymptotic lines is given by the differential equation $2Mdu+Ndv=0$, or

$$\frac{du}{dv}+\frac{\lambda}{2\delta}+\frac{\mu}{2\delta}\cdot u+\frac{\nu}{2\delta}\cdot u^2=0,$$

it being assumed that $\delta\neq 0$ so that the ruled surface is non-developable.

This is a Riccati's equation and cannot be integrated in finite terms for all values of λ, μ, ν, δ, i.e., for all ruled surfaces. If, however, one particular solution of Riccati's equation is known, all others can be found by quadratures.[6]

We shall examine certain conditions under which this Riccati's equation can be integrated.

3. I. If $\nu\equiv\sum l\,(m'\,n''-n'\,m'')=0$, i.e., if the generators of the ruled surface remain parallel to a fixed plane, the Riccati's equation reduces to the linear equation $\frac{du}{dv}+\frac{\lambda}{2\delta}+\frac{\mu}{2\delta}\cdot u=0$, and asymptotic lines can be determined by two quadratures.

Taking the fixed plane as the plane of xy, the equation of the surface can be written, without loss of generality, in the form $x=u$, $y=v+f(v)\,u$, $z=F(v)$.

Here $\delta\equiv\sum l\,(q'\,n'-r'\,m')=-f'\,F'$,

$$\mu\equiv\sum l\,\{(m'\,r''-m''\,r')+(q'\,n''-n'\,q'')\}=(f'\,F''-f''\,F'),$$

$$\lambda\equiv\sum l\,(q'\,r''-r'\,q'')=F''.$$

$\therefore$ the equation giving the curved asymptotic lines becomes

$$\frac{du}{dv}-\frac{F''}{2f'\,F'}-\frac{1}{2f'\,F'}(f'\,F''-f''\,F')\,u=0,$$

whose solution is

$$u=\sqrt{\frac{F'}{f'}}\left(\int\frac{F''}{2\sqrt{f'\,F'^3}}\,dv+\text{const.}\right).$$

Two cases present themselves in which the asymptotic lines can be determined in finite form

(i) Let $f' = \text{const} = c_1$ (say), so that $f = c_1 v + c_2$.

The equations of the surface are of the form

$$x = u, \quad y = v + (c_1 v + c_2) u, \quad z = F(v).$$

All the generators meet $y - c_2 x = 0$, $1 + c_1 x = 0$.

The equation of the asymptotic lines reduces to $u = \text{const}$.

(ii) Let $c_3 f' = F'$ where c_3 is a constant, so that $F = c_3 f + c_4$.

The equations of the surface are of the form

$$x = u, \quad y = v + uf(v), \quad z = c_3 f(v) + c_4.$$

The equation of the asymptotic lines takes the form

$$u = -\frac{c_3}{2F'} + \text{const}.$$

II. If $\lambda \equiv \sum l\,(q' r'' - r' q'') = 0$, i.e., if the generators are perpendicular to the binormals of the directrix curve, which implies that the base curve is an asymptotic, the Riccati's equation reduces to

$$\frac{du}{dv} + \frac{\mu}{2\delta} u + \frac{\nu}{2\delta} u^2 = 0,$$

which is Bernoulli's equation whose solution is

$$\frac{1}{u} \cdot e^{-\int \frac{\mu}{2\delta} dv} = \int e^{-\int \frac{\mu}{2\delta} dv} \cdot \frac{\nu}{2\delta} dv + \text{const}.$$

The condition $\lambda = 0$ is satisfied by all surfaces in whose equations any two of the three quantities p, q, r are either constants or zero. The most general form of the equation of such surfaces is

$$x = k_1 + u, \quad y = k_2 + u f(v), \quad z = \phi(v) + u\,\psi(v).$$

Remarks. (*a*) A special case of such ruled surfaces which has been considered by Srinivasiengar[7] is a surface of degree n which possesses a multiple line of the $(n-1)$th order.

The equation of such a surface can be written in the parametric form $x = u$, $y = uv$, $z = f(v) + u\phi(v)$.

Here $\delta=-f'(v)$, $\lambda=0$, $\mu=f''(v)$, $\nu=\phi''(v)$.

$\therefore$ the Riccati's equation reduces to

$$\frac{du}{dv}-\frac{f''(v)}{2f'(v)}u-\frac{\phi''(v)}{2f'(v)}u^2=0,$$

whose solution comes out to be

$$\frac{1}{u}\cdot[f'(v)]^{\frac{1}{2}}=\int[f'(v)]^{\frac{1}{2}}\cdot\frac{\phi''(v)}{-2f'(v)}\,dv+c,$$

and can be obtained by quadratures.

(*b*). This case also includes all ruled surfaces whose line of striction is straight.

For if this straight line be taken as the directrix and be chosen as the axis of z, we can put $p=0$, $q=0$, $r=v$, and, therefore the equations of a ruled surface which has straight line of striction are of the form

$$x=lu,\quad y=mu,\quad z=v+u\cos\theta,$$

where θ is the constant angle which the generators make with the straight line of striction. These equations can be made to represent a Helicoid or a Right Conoid.[8]

III. If $\lambda=0$ and $\nu=0$, the Riccati's equation reduces to $\frac{du}{dv}+\frac{\mu u}{2\delta}=0$ whose solution is $u=c\cdot e^{-\int\frac{\mu}{2\delta}dv}$, so that the curved asymptotic lines can be determined by one quadrature only.

This is the case when two asymptotic lines are known.

IV. If $\lambda=0$, $\nu=0$, $\mu=0$, the Riccati's equation reduces to $\frac{du}{dv}=0$ which gives $u=\text{const.}$

This case includes the right helicoids whose equations, we know, are of the form $x=u\cos v$, $y=u\sin v$, $z=k+cv$.

N.B. Since $D\equiv lp'=0$, the orthogonal trajectories of the generators $v=\text{const.}$ are given by $u+\int D\,dv=\text{const.}$, i.e., by $u=\text{const.}$, which gives the known result that "the curved asymptotic lines of a right helicoid are the orthogonal trajectories of the generators."

References.

1. Annales de L'ecole *Normale*, *(1877)*, p. 346; or Traite d'analyse, t. 1, p. 400.

2. Nouvelles Annales de Mathematiques, (1908), pp. 340-346; Liouville, Journal de Mathemat., (1929), pp. 48, 51-52.

3. Cours d'analyse Mathematique t. 1, p. 605.

4. Jour. Ind. Math. Soc. (1931), pp. 44-48; or Zentralblatt fur Mathematik, vol. 2, (1932), p. 286.

5. Sc. Rep. Tohoku Imp. Univ. (1917), pp. 5-6.

6. Forsyth, A Treatise on Differential Equations 5th ed., p. 192; also Bour, Journal d'ecole polytechnique (1862), p. 48.

7. loc. cit.

8. Hayashi, loc. cit., pp. 5-6.

INFINITESIMAL AUTOMORPHISMS OF THE ACTION FORM

BY

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Closely associated with the Lagrangian is the fundamentally important Pfaffian form

$$A_d = p_r dq^r - H dt$$

which may be called the *Action form*. (The repeated index stands for summation from 1 to n except when otherwise mentioned.) This form, taken to be of class $2n+1$ in the $2n+1$ variables (q, p, t) is an epitome of most of the dynamical concepts* when examined from the class-theoretic point of view, while the motion of the dynamical system associated with A_d can be regarded as the 'gradual self unfolding '† of a suitable contact transformation of A_d. The general infinitesimal contact transformations of A_d include as a subgroup the infinitesimal automorphisms of A_d transforming $p_r dq^r - H dt$ into $P_r dQ^r - H' dT$.

The following are some of the interesting questions that arise:

(i) can the infinitesimal automorphisms of A_d be derived as a special case from the solution of the general infinitesimal contact transformations of A_d?,

(ii) what is the necessary and sufficient condition for the Hamiltonian H also to admit the infinitesimal automorphisms of A_d?,

and (iii) is it possible for the transformation paths to include dynamical trajectories which are the integral curves of A_d?

The object of the present paper is to answer these questions.

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* K. Nagabhushanam, On the form $\Sigma\, p_r\, dq^r - Hdt$. *Proc. Ind. Ac. Sci.* Bangalore, Vol. I, No. 8., (1935).

† E. T. Whittaker: *Analytical Dynamics*, 3rd edn. p. 304. The infinitesimal contact transformations whose gradual unfoldment represents the motion of the dynamical system have been studied by the author as I. T. transformations in his paper on 'The Transformation Theory of Dynamics in the manifold of States and Time'. *Journal. Ind. Math. Soc.* Vol. 20, p. 238.

The general contact transformations of A_d are given in terms of the arbitrary generating function W, if A_d is changed into $A_d + dW$. In the present case of infinitesimal automorphisms of A_d, $dW = 0$, with the result that there is no differential equation for W. Hence it is evident that the infinitesimal automorphisms of A_d should not be sought as a particular case of the general infinitesimal contact transformations of A_d.

The results of this paper show that H admits the infinitesimal automorphisms of A_d if and only if $\frac{\partial H}{\partial t} = 0$.

Again class theoretic considerations make it clear that the paths of infinitesimal automorphisms of A_d cannot include the dynamical trajectories.

Infinitesimal automorphisms of A_d

The method suited for the present inquiry is due to L. P. Eisenhart.* The solution of his homogeneous contact transformations forms a special case of the solution here.

Let the infinitesimal automorphisms be denoted by

$$\delta q^r = \phi\xi^r\,\delta\tau,\ \delta p_r = \varphi\eta_r\,\delta\tau,\ (r = 1, 2. \ldots n);\ \delta t = \varphi\zeta\delta\tau,$$

where φ is arbitrary. Then

$$\delta A_d = (p_r + \varphi\eta_r\,\delta\tau)\,d\,(q^r + \varphi\xi^r\,\delta\tau) - p_r\,dq^r$$
$$\left(-H + \frac{\partial H}{\partial q^r}\varphi\xi^r\,\delta\tau + \frac{\partial H}{\partial p_r}\varphi\eta_r\,\delta\tau + \frac{\partial H}{\partial t}\varphi\zeta\delta\tau\right) d\,(t + \varphi\xi\delta\tau)$$
$$+ H dt.$$

Assuming that $d\delta\tau = 0$, and equating to zero the coefficients of $\varphi\ \delta\tau\ dq^k$, $\varphi\ \delta\tau\ dp_k$, $\varphi\ \delta\tau\ dt$, $d\varphi\delta\tau$ and neglecting small quantities of higher order, we have

$$\eta_k + p_r\frac{\partial\xi^r}{\partial q^k} - H\frac{\partial\zeta}{\partial q^k} = 0 \qquad \ldots \quad (1),$$

$$p_r\frac{\partial\xi^r}{\partial p_k} - H\frac{\partial\zeta}{\partial p^k} = 0 \qquad \ldots \quad (2),$$

$$p_r\xi^r - H\zeta = 0 \qquad \ldots \quad (3),$$

$$p_r\frac{\partial\xi^r}{\partial t} - H\frac{\partial\zeta}{\partial t} - \frac{\partial H}{\partial t}\zeta - \frac{\partial H}{\partial q^r}\xi^r - \frac{\partial H}{\partial p_r}\eta_r = 0 \qquad \ldots \quad (4).$$

* *Contact Transformations*, Annals of Mathematics, Second Series, Vol. 30, No. 2, p. 224.

If we put $p_r \xi^r = f$, we have

$$p_r \frac{\partial \xi^r}{\partial q^k} = \frac{\partial f}{\partial q^k}; \quad \xi^k + p_r \frac{\partial \xi_r}{\partial p_k} = \frac{\partial f}{\partial p_k}; \quad p_r \frac{\partial \xi^r}{\partial t} = \frac{\partial f}{\partial t}.$$

The equations (1), (2), (3), (4) becomes

$$\eta_k = -\frac{\partial f}{\partial q^k} + H\frac{\partial \zeta}{\partial q^k} \qquad \ldots \quad (1'),$$

$$\xi^k = \frac{\partial f}{\partial p_k} - H\frac{\partial \zeta}{\partial p_k} \qquad \ldots \quad (2'),$$

$$f - H\zeta = 0 \qquad \ldots \quad (3'),$$

so that

$$\frac{\partial f}{\partial t} - \frac{\partial H}{\partial t}\zeta - H\frac{\partial \zeta}{\partial t} = 0 \qquad \ldots \quad (3'a)$$

and

$$\frac{\partial H}{\partial q^r}\xi^r + \frac{\partial H}{\partial p_r}\eta_r = 0 \qquad \ldots \quad (4')$$

by virtue of $(3'a)$.

It is easily seen by direct calculation that (4′) is a consequence of (1′), (2′) and (3′). For

$$\frac{\partial H}{\partial q^r}\xi^r + \frac{\partial H}{\partial p_r}\eta_r = \frac{\partial H}{\partial q^r}\left(\frac{\partial f}{\partial p_r} - H\frac{\partial \zeta}{\partial p_r}\right) + \left(-\frac{\partial f}{\partial q^r} + H\frac{\partial \zeta}{\partial q^r}\right)$$

$$= \frac{\partial H}{\partial q^r}\left(\frac{\partial f}{\partial p_r} - \frac{\partial H}{\partial p_r}\zeta - H\frac{\partial \zeta}{\partial p_r}\right)$$

$$+ \frac{\partial H}{\partial p_r}\left(-\frac{\partial f}{\partial q^r} + \frac{\partial H}{\partial q^r}\zeta + H\frac{\partial \zeta}{\partial q^r}\right)$$

$$= 0, \text{ since } \left.\begin{array}{l} \dfrac{\partial}{\partial q^r}(f - H\zeta) = 0 \\ \text{and } \dfrac{\partial}{\partial p_r}(f - H\zeta) = 0 \end{array}\right\} \text{by } (3').$$

Hence we have

THEOREM 1.—*If f and ζ are two functions satisfying the relation $f - H\zeta = 0$, the equations* (1′) *and* (2′) *give the most general solution of the infinitesimal automorphisms of A_d.*

If f is homogeneous of degree one in the p's, and ζ an absolute constant, the solutions of ξ^k, η_k give the homogeneous contact transformation of Eisenhart.

Conditions under which H admits the infinitesimal automorphisms of A_d

$$\delta H = \frac{\partial H}{\partial q^r}\varphi\xi^r\delta\tau + \frac{\partial H}{\partial p_r}\varphi\,\eta_r\delta\tau + \frac{\partial H}{\partial t}\varphi\zeta\delta\tau$$

$$= \varphi\frac{\partial H}{\partial t}\zeta\delta\tau \qquad \text{by (4')}$$

Hence if $\delta H = 0$, then $\frac{\partial H}{\partial t} = 0$, since $\varphi\zeta\delta\tau$ is arbitrary. Also $\delta H = 0$ if $\frac{\partial H}{\partial t} = 0$. Thus we have

THEOREM 2.—*The condition that H is explicitly independent of t is both necessary and sufficient for H to admit the infinitesimal automorphisms of A_d.*

We shall now establish the following result:

THEOREM 3. *The paths of infinitesimal automorphisms of A_d cannot include the dynamical trajectories.*

In this last section, A_d is written as $X_i dx^i$, where i runs through the indices $1, 2, \cdots 2n+1$, and the class of $X_i\,dx^i$, viz., the rank of the matrix $|a_{ik};\, X_k|$ where $\left(a_{ik} = \frac{\partial X_i}{\partial x^k} - \frac{\partial X_k}{\partial x^i}\right)$, is $2n+1$. Let the infinitesimal automorphisms of A_d be now denoted by

$$\delta x^i = \phi\ \sigma^i\ \delta\tau, \qquad i = 1, 2 \cdots 2n+1.$$

$$\delta A_d = \left(X_i + \frac{\partial X_i}{\partial x^k}\varphi\sigma^k\delta\tau\right)d(x^i + \varphi\sigma^i\delta\tau) - X_i\,dx^i.$$

Equating to zero, the co-efficients of $\varphi dx^i\delta\tau$, and $d\varphi\delta\tau$, we obtain

$$\left.\begin{aligned} &\frac{\partial X_i}{\partial x^k}\sigma^k + \frac{\partial\sigma^k}{\partial x^i}X_k = 0, \qquad (i = 1, 2 \cdots 2n+1) \\ \text{and}\quad &X_i\,\sigma^i = 0 \end{aligned}\right\} \qquad \cdots (5)$$

If the trajectories are to be included in the paths of the transformation, σ^i must satisfy the equations (5) whenever $a_{ik}\,\sigma^k = 0$.

Hence $X_k\,\sigma^k = 0$ is to be a consequence of the set $a_{ik}\sigma^k = 0$, with the result that the rank of the matrix $|a_{ik};\, X_k|$ is to be the same as that of $|a_{ik}|$ viz. $2n$ or less. Thus the class of A_d will have to be $\leqslant 2n$. As this is impossible, the trajectories cannot be included in the paths of the transformation.

ON THE VOLUME OF A PRISMOID IN N—SPACE AND SOME PROBLEMS IN CONTINUOUS PROBABILITY.

BY

C. RADHAKRISHNA RAO.

Let $x_1, x_2 \cdots\cdots x_n$ be n independent variables. An ordered set $(x_1, x_2 \cdots\cdots\cdots x_n)$ can be said to represent a point in n dimensional space. A linear equation of these variables such as $l_1 x_1 + l_2 x_2 + \cdots\cdots\cdots = k$ represents a hyperplane in n dimensions.

The hyperplanes
$$\begin{array}{ll} x_1 = a_1, & x_1 = b_1, \\ x_2 = a_2, & x_2 = b_2, \\ \cdots\cdots & \cdots\cdots \\ x_n = a_n, & x_n = b_n, \end{array}$$

enclose a certain space D called the Prismoid.

§1. *To find the volume either exterior or interior to the prismoid cut off by the hyperplane*

$$l_1 x_1 + l_2 x_2 \cdots\cdots = k.$$

Starting with a simple case let it be required to find the volume enclosed by

$$\left.\begin{array}{ll} (x_1 > c_1,) & 0 \leqslant x_{i+1} \leqslant c_{i+1} \\ (x_2 > c_2,) & \cdots\cdots\cdots \\ (\cdots\cdots) & 0 \leqslant x_n \leqslant c_n \\ (x_i > c_i,) & 0 \leqslant S(x_n) \leqslant P \end{array}\right\} \qquad (1\cdot1)$$

where $S(x_n)$ means $x_1 + x_2 \cdots\cdots + x_n$.

Obviously $P > c_1 + c_2 \cdots\cdots\cdots c_i$.

This volume is the same as that enclosed by

$$\begin{array}{ll} (y_1 > 0) & 0 \leqslant y_{i+1} \leqslant c_{i+1} \\ (y_2 > 0) & 0 \leqslant y_n \leqslant c_n \\ (\cdots\cdots) & 0 \leqslant S(y_n) \leqslant P - (c_1 + c_2 \cdots\cdots + c_i) = Q \\ (y_i > 0) & \end{array}$$

in a different space, got by the substitutions

$$x_1 - c_1 = y_1, \qquad x_i - c_i = y_i$$

$$x_2 - c_2 = y_2, \qquad x_{i+1} = y_{i+1}$$

$$\cdots\cdots \qquad x_n = y_n$$

The volume of the region bounded by the coordinate planes and the plane $S(y_n) = Q$ is given* by $\dfrac{Q^n}{n!}$ (A)

Let $Q > c_s \quad s = h, u, v, \cdots\cdots$

Then the above expression includes the volume above certain faces

$y_s = c_s \quad s = h, u, v \cdots\cdots$

of the domain D. The sum of the different volumes above these faces is

$$\frac{(Q - c_s)^n}{n!} + \frac{(Q - c_u)^n}{n!}, \cdots \qquad \text{(B)}$$

where $Q - c_r$ is greater than zero.

In case $Q > c_h + c_r$, the volume exterior to both planes $y_h = c_h$, $y_r = c_r$ has been included in (A);

This volume has been already subtracted twice, once in $\dfrac{(Q - c_h)^n}{n!}$ and once in $\dfrac{(Q - c_r)^n}{n!}$. So the volume $\dfrac{(Q - c_h - c_r)^n}{n!}$ above the two faces and enclosed by the above plane has to be added to (A) − (B).

Let the sum of similar terms be

$$\frac{(Q - c_u - c_r)^n}{n!} + \frac{(Q - c_v - c_t)^n}{n!} + \cdots \qquad \text{(C)}$$

the bracket in each case being positive.

Similarly we can form the sums D, E, F .... and the volume is easily seen to be

$$A - B + C - D \cdots\cdots \qquad (1\cdot 2)$$

The result for conditions (1·1) is obtained by writing

$$Q = P - (c_1 + c_2 \cdots + c_i)$$

* The value of the Dirichlet's integral.

If the conditions are $0 \leqslant x_r \leqslant c_r$ $(r=1 \cdot 2 \cdots n)$ then clearly the result becomes

$$\frac{1}{n!}\{P^n - S(P-c_s)^n + S(P-c_t-c_u)^n - \cdots\}$$

$$P-c_s > 0, \ (P-c_t-c_u) > 0, \text{ etc.} \qquad (1\cdot3)$$

The volume of the lower portion of the domain D formed by

$$x_r = a_r, \ x_r = b_r, \ (r=1, 2 \cdots n)$$

cut off by the hyperplane is given by writing

$$l_1 x_1 + l_2 x_2 \cdots = k$$

$$P = k - S(a_r l_r); \qquad c_r = l_r (b_r - a_r)$$

in (1·3) and multiplying it by $(l_1 l_2 \cdots l_n)^{-1}$, the result being obtained after suitable substitutions $l_r (x_r - a_r) = y_r$.

Incidentally, this gives the proportionate probability of the occurrence of values less than a certain magnitude for the mean of a number of observations each drawn from rectangular distributions with a finite range.

If all the c's are equal the results (1·2) and (1·3) become

$$(P-ic)^n - {}_{n-1}C_1 (P-\overline{i+1}\,c)^n + {}_{n-i}C_2 \ (P-\overline{i+2}\,c)^n$$

$$\cdots (-1)^k \, {}_{n-i}C_k (P-\overline{i+k}\,c)^n \qquad (1\cdot22)$$

where k is such that

$$(i+k)c \leqslant P < (i+k+l)c$$

and $\frac{1}{n!}\{P^n - {}_nC_1 (P-c)^n + {}_nC_2 (P-2c)^n \cdots (-1)^r \, {}_nC_r (P-rc)^n\}$ (1·33)

r is such that $rc \leqslant p < (r+1)c$

§2. On the circumference of a circle of unit length, n arcs of lengths $h_1, h_2 \cdots h_n$ are marked off in a random order. What are the probabilities that

1. there are r gaps;
2. there are no gaps;
3. the intervals do not overlap by a distance more than c.

W. L. Stevens* gave the result for 1 and 2 when $h_1 = h_2 \cdots$ as a deduction from a general theorem.

Let the intervals be $(0, h_1)$, (x_1, x_1+h_2), (x_2, x_2+h_3), $\cdots$ $(x_{n-1}, x_{n-1}+h_n)$ where $0 < x_1 < \cdots x_{n-1}$. If there be r gaps after the intervals of length h_{r+1}, h_{s+1} etc. but nowhere else we have

$$\begin{array}{ll} 0 < x_1 < h_1 & x_{r+1} - x_r > h_{r+1} \\ 0 < x_2 - x_1 < h_2 & x_{s+1} - x_s > h_{s+1} \\ \cdots\cdots & \cdots\cdots \end{array}$$

and finally $0 \leqslant 1 - x_{n-1} \leqslant h_n$ since there is no gap after the interval h_n

The inequalities become after substitutions $x_1 = y_1$, $x_2 - x_1 = y_2$, etc.

$$\left.\begin{array}{ll} 1 - h_n \leqslant S(y_{n-1}) \leqslant 1 & \\ 0 \leqslant y_1 \leqslant h_1 & y_{r+1} > h_{s+1} \\ \cdots\cdots & \cdots\cdots \\ 0 \leqslant y_{k+1} \leqslant h_{k+1} & y_{u+1} > h_{u+1}. \end{array}\right\} \qquad (b)$$

To find the probability for this specified arrangement we have to find the volume enclosed by the above hyperplanes viz. the boundaries of $S(y_{n-1})$ in the domain defined by (b) and divide it by the total volume of the domain of freedom of $y_1\ y_2 \ldots y_{n-1}$ which is equal to 1.

Employing (1·2) we get

$$k = \frac{1}{(n-1)!}\ \{(P^{n-1} - S(P-h_1)^{n-1} + S(P-h_x-h_s)^{n-1} \ldots)\}$$

$$= \frac{1}{(n-1)!}\ \{(P-h_n)^{n-1} - S(P-h_n-h_t)^{n-1} \ldots\ldots\ldots)\}$$

where $P = 1 - h_{r+1} \ldots\ldots\ldots - h_{u+1}$

Hence $K = \dfrac{1}{(n-1)!}\ \{(P^{n-1} - S(P-h_1)^{n-1} + \ldots\ldots\ldots)\}$

where h_n is included in the summations, a symmetrical result which does not depend on the origin chosen.

Since there are $(n-1)!$ arrangements of these intervals round the circle we get the probability for the specified arrangement

$$P_{r\ldots u} = (n-1)!\ k$$

where $P_{r\ldots u}$ denotes the probability for gaps after $r+1, \cdots (u+1)$th intervals.

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* Annals of Eugenics 1939.

The compound probability for any r gaps is given by

$$S(P_{r \ldots n}) \tag{2·1}$$

The summation consisting of ${}_nC_r$ terms.

When there are no gaps the inequalities satisfy (1·3) and hence by similar reasoning we get

$$P_0 = 1 - S(1-h_r)^{n-1} + S(1-h_r-h_s)^{n-1} \cdot \cdot \cdot \cdot \tag{2·2}$$

In particular if $h_1 = h_2 \cdot \cdot \cdot \cdot = x$ the results become

$$P_r = {}_nC_r \{(1-rx)^{n-1} - {}_{n-r}C_1 (1-x)^{n-1} + \cdot \cdot \cdot\} \tag{2·3}$$

$$P_0 = \{1 - {}_nC_1 (1-x)^{n-1} + {}_nC_2 (1-2x)^{n-1} \cdot \cdot \cdot \cdot \cdot \cdot (-1)^k {}_nC_k (1-kx)^{n-1}\}$$

$$kx \leqslant 1 < (k+1)\, x$$

The mean number of gaps is easily seen* to be $n(1-x)^{n-1}$

To solve 3 we have the following inequalities

$$(x_1 > h_1 - c) \qquad x_{n-1} - x_{n-2} > h_{n-1} - c$$

$$(x_2 - x_1 > h_2 - c) \qquad 1 - x_n + c > h_n$$

$$\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

By the usual substitution we get

$$y_r > k_r, \qquad (r = 1, \cdot \cdot \cdot \cdot \cdot \cdot \; n-1),$$

and $k_r = h_r - c$, and $S(y_r) \leqslant 1 - k_n$

This leads us to find the volume exterior to be prismoid bounded by the hyperplane $S(y_r) = 1 - k_n$

$$= \frac{(1 - k_n - k_2 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot - k_1)^{n-1}}{(n-1)!}$$

$$= \frac{(1 - h_1 - h_2 \cdot \cdot \cdot \cdot \cdot - h_n + nc)^{n-1}}{(n-1)!}$$ provided the expression

within the brackets is positive.

So the probability is given by $\{1 + nc - S(h_r)\}^{n-1}$

* Discussed in the end.

If in addition we have no gaps the inequalities become

$$h_1 > x_1 > h_1 - c$$

$$h_2 > x_2 - x_1 > h_2 - c$$

$$\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

and with the boundary conditions $h_n > 1 - x_{n-1} > h_n - c$.

This again reduces to the above case on making the substitutions

$$y_1 = h_1 - x_1$$

$$y_2 = h_2 - x_2 + x_1$$

$$\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

The boundary condition becomes

$$h_n > 1 + S(y_r) - S(h_r) > h_n - c$$

or

$$-1 + S(h_r) > S(y_r) > -1 - c + S(h_r)$$

h_n included $\qquad$ h_n included

and the probability can be easily given by writing $S(h_r) - 1 - c$ and $S(h_r) - 1$ for P in (1·3) and subtracting one from the other.

In the case of a straight line the boundary condition changes. For instance, if segments of lengths, $h_1, h_2 \cdots h_n$ are marked off at random on a line of length 1 what is the probability that there are no gaps, and yet they do not overlap by a length c?

O h_1 | $\qquad$ | h_n A

x_1 $\qquad$ x_{n-1}

$$h_1 - c \leqslant x_1 \leqslant h_1$$

$$h_2 - c \leqslant x_2 - x_1 \leqslant h_2$$

$$\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

$$h_{n-2} - c \leqslant (x_{n-2} - x_{n-3}) \leqslant h_{n-2}$$

and $\quad 1 - x_{n-2} - h_{n-1} > h_n - c$

$$h_n + h_{n-1} > 1 - x_{n-2} > h_n + h_{n-1} - c$$

This again leads to the one of the sets of inequalities and the corresponding probability can be written. With similar considerations many problems in continuous probability can be solved.

3. The distribution of the number of gaps

Putting ${}_nC_r\ (1-rx)^{n-1}=x_r$ we have if P_m denotes the probability for m gaps,

$$P_0=1-x_1+x_2-x_3 \cdot\cdot\cdot\cdot\cdot (-1)^k x_k$$

$$P_1=x_1-2x_2+3x_3 \cdot\cdot\cdot\cdot\cdot\cdot (-1)^{k-1}\ kx_k$$

$$P_2=x_2-{}_3C_2\ x_3 \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot (-1)^{k-2}\ {}_kC_2\ x_k$$

$$P_k= \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot +x_k$$

The moments of this distribution are

$$\mu_1'=x_1\ ;\ \mu_2'=x_1+2x_2\ ;\ \mu_3'=x_1+3\ !\ (x_2+x_3)$$

$$\mu_4'=x_1+14x_2+36x_3+4\ !\ x_4$$

μ's can also be written in a similar manner although the general expression is complicated.

$$\mu_1=0\ ;\ \mu_2=x_1+2x_2-x_1^2=x_1\ (1-x_1)+2x_2$$

The distribution is skew, except when $\mu_3=0$;

$$\text{i.e., } x_1+3\ !\ (x_2+x_3)+2x_1^3=3x_1\ (x_1+2x_2)$$

HOWLERS

Algebra is the wife of Euclid.

A triangle is a straight line bent twice so that the two ends meet.

THE EVALUATION OF CERTAIN DETERMINANTS

BY

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1. In this paper I evaluate certain determinants whose elements are trigonometric functions.

Let $f(x)$ denote the function

$$\sum_{l=1}^{r} a_l \exp. (m_l x)$$

Then, if we denote the determinant of the nth order whose element in the ith row and jth column is $a_{i,j}$ by $D_n[a_{i,j}]$, we have

THEOREM 1.

$$D_n[f(\theta_i + \varphi_j)] = 0 \text{ if } n > r.$$

Put exp. $(m_k \theta_i) = x_{k,i}$ and exp. $(m_k \varphi_i) = y_{k,i}$. Then

$$D_n [f(\theta_i + \varphi_j)] = D_n\Big[\sum_{l=1}^{r} a_l x_{l,i} y_{l,j} \Big]$$

$$= \prod_{i=1}^{n} \Big\{ \sum_{j=1}^{n} \varepsilon_j \Big(\sum_{l=1}^{r} a_l x_{l,i} y_{l,j} \Big) \Big\},$$

where ε_j are the usual symbols (alternating units) occurring in the theory of determinants, satisfying the properties

$$\Pi\, \varepsilon_j = 1, \quad \varepsilon_i \varepsilon_j = -\varepsilon_j \varepsilon_i, \quad \varepsilon_i^2 = 0.$$

Therefore

$$D_n[f(\theta_i + \varphi_j)] = \prod_{i=1}^{n} \sum_{l=1}^{r} a_l x_{l,i} A_l,$$

where A_l is the alternating number $\sum_{j=1}^{n} \varepsilon_j y_{l,j}$.

On multiplying out the above product we see that each term has n alternating numbers A_l as factors. If $n > r$, since there are only r distinct alternating numbers A_l, there will be repeated factors in each term. Therefore from the fact that the square of an alternating number is zero, it follows that all the terms vanish. Thus we get the required result.

COROLLARY*. $D_n[\sin(\theta_i+\varphi_j)] = D_n[\cos(\theta_i+\varphi_j)] = 0$ if $n \geq 3$.

2. We require the following lemmas:

LEMMA 1.

$$D_n\left[\frac{1}{x_i+y_j}\right] = \prod_{\substack{i,j=1\\(i>j)}}^{n} (x_i-x_j)(y_i-y_j) \prod_{i,j=1}^{n} 1/(x_i+y_j).$$

PROOF: $D_n\left[\frac{1}{x_i+y_j}\right]$ vanishes whenever $x_i=x_j$, $y_i=y_j$ $(i \neq j)$.

Therefore $\Pi\,(x_i-x_j)(y_i-y_j)$ is a factor of $D_n\left[\frac{1}{x_i+y_j}\right]$.

But when the determinant $D_n\left[\frac{1}{x_i+y_j}\right]$ is expanded, and the terms put on a common denominator which is simply the product of all the denominators of the elements of the determinant, the numerator will be a homogeneous expression of degree $n(n-1)$ in $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$.

It follows that

$$D_n\left[\frac{1}{x_i+y_j}\right] = k \prod_{\substack{i,j=1\\(i>j)}}^{n} (x_i-x_j)(y_i-y_j) \div \prod_{i,j=1}^{n} (x_i+y_j),$$

where k is some constant. Comparing the co-efficients of $x_2x_3^2 \cdots x_n^{n-1}y_2y_3^2 \cdots y_n^{n-1}$ in the numerators on the two sides we see that $k=1$. Thus the lemma is proved.

LEMMA 2. $D_n\left[\frac{x_i-y_j}{x_i+y}\right]$

$$= 2^{n-1}(x_1x_2\cdots x_n + (-1)^n y_1y_2\cdots y_n) \prod_{\substack{i,j=1\\(i>j)}}^{n} (x_i-x_j)(y_i-y_j) \times \prod_{i,j=1}^{n} \frac{1}{x_i+y_j}.$$

PROOF: It is easily seen that

$$D_n\left[\frac{x_i-y_j}{x_i+y_j}\right] = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \frac{2y_1}{x_1+y_1} & \frac{2y_2}{x_1+y_2} & \cdots & \frac{2y_n}{x_1+y_n} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{2y_1}{x_n+y_1} & \frac{2y_2}{x_n+y_2} & \cdots & \frac{2y_n}{x_n+y_n} & 1 \end{vmatrix}$$

* This may also be proved by making the substitutions $\tan\theta_i = x_i$ and $\tan\varphi_i = y_i$ and noting that $D_n[x_i+y_j] = 0$.

$$= 2^{n-1} y_1 y_2 \cdots y_n \begin{vmatrix} \frac{1}{y_1} & \frac{1}{y_2} & \cdots & \frac{1}{y_n} & 2 \\ \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{1}{x_n+y_1} & \frac{1}{x_n+y_2} & \cdots & \frac{1}{x_n+y_n} & 1 \end{vmatrix} = 2^{n-1} y_1 y_2 \cdots y_n D', \text{ (say)}.$$

Now
$$D' = \begin{vmatrix} \frac{1}{y_1} & \frac{1}{y_2} & \cdots & \frac{1}{y_n} & 1 \\ \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{1}{x_n+y_1} & \frac{1}{x_n+y_2} & \cdots & \frac{1}{x_n+y_n} & 1 \end{vmatrix} + (-1)^n D_n\left[\frac{1}{x_i+y_j}\right]$$

But by lemma 1, $D_n\left[\frac{1}{x_i+y_j}\right] = \Pi\ (x_i - x_j)(y_i - y_j)\ \Pi\ \frac{1}{(x_i+y_j)}$

In a similar way we can prove that

$$\begin{vmatrix} \frac{1}{y_1} & \frac{1}{y_2} & \cdots & \frac{1}{y_n} & 1 \\ \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{1}{x_n+y_1} & \frac{1}{x_n+y_2} & \cdots & \frac{1}{x_n+y_n} & 1 \end{vmatrix} = \frac{x_1 x_2 \cdots x_n}{y_1 y_2 \cdots y_n} \prod_{\substack{i,j=1 \\ i>j}}^{n} (x_i - x_j)(y_i - y_j) \times \prod_{i,j=1}^{n} \frac{1}{x_i + y_j}.$$

Thus we get $D_n\left[\frac{x_i - y_j}{x_i + y_j}\right] =$

$$2^{n-1}(x_1 x_2 \cdots x_n + (-)^n y_1 y_2 \cdots y_n) \prod^{n} (x_i - y_j)(y_i - y_j) \times \prod_{i,j=1}^{n} \frac{1}{(x_i + y_j)}.$$

THEOREM 2.

$$D_n\,[\operatorname{cosec}\,(\theta_i + \varphi_j)] = \prod_{\substack{i,j=1 \\ (i>j)}}^{n} \sin\,(\theta_i - \theta_j)\,\sin\,(\varphi_i - \varphi_j) \prod_{i,j=1}^{n} \operatorname{cosec}\,(\theta_i + \varphi_j).$$

Put $\tan\theta_i = x_i$ and $\tan\varphi_i = y_i$.

$$\text{Then } D_n\,[\operatorname{cosec}(\theta_i+\varphi_j)] = D_n\left[\frac{1}{x_i+y_j}\right]\prod_{i=1}^{n}\sqrt{(1+x_i^2)(1+y_i^2)}.$$

$$= \prod_{\substack{i,j=1\\(i>j)}}^{n}(x_i-x_j)(y_i-y_j)\prod_{i,j=1}^{n}\frac{1}{x_i+y_j}\prod_{i=1}^{n}\sqrt{(1+x_i^2)(1+y_i^2)}$$

by lemma 1.

$$= \prod_{\substack{i,j=1\\i>j}}^{n}\frac{(x_i-x_j)}{\sqrt{(1+x_i^2)(1+x_j^2)}}\,\frac{(y_i-y_j)}{\sqrt{(1+y_i^2)(1+y_j^2)}}\prod_{i,j=1}^{n}\frac{\sqrt{(1+x_i^2)(1+y_i^2)}}{x_i+y_j}.$$

Substituting the values of x_i and y_i we get the required result.

COROLLARY. Putting $\frac{1}{2}\pi+\theta_i$ instead of θ_i we get

$$D_n\,[\sec(\theta_i+\varphi_j)] = \prod_{\substack{i,j=1\\(i>j)}}^{n}\sin(\theta_i-\theta_j)\sin(\varphi_i-\varphi_j)\prod_{i,j=1}^{n}\sec(\theta_i+\varphi_j).$$

THEOREM 3.

$$D_n\left[\frac{\sin(\theta_i-\varphi_j)}{\sin(\theta_i+\varphi_j)}\right] = \frac{2^{n-1}\left(\prod_{i=1}^{n}\tan\theta_i+(-1)^n\prod_{i=1}^{n}\tan\phi_i\right)}{\sec\theta_1\sec\theta_2\cdots\sec\theta_n\cdot\sec\varphi_1\sec\varphi_2\cdots\sec\varphi_n}.$$

$$\times\prod_{\substack{i,j=1\\(i>j)}}^{n}\sin(\theta_i-\theta_j)\sin(\varphi_i-\varphi_j)\prod_{i,j=1}^{n}\operatorname{cosec}(\theta_i+\varphi_j)$$

This can be proved like theorem 2 with the help of lemma 2

Replacing θ_i by $(\frac{1}{2}\pi+\theta_i)$, θ_i by $\pi/4+\theta_i$ and φ_i by $(\pi/4+\varphi_i)$ and θ_i by $\pi/4+\theta_i$ and φ_i by $(\varphi_i-\pi/4)$ we can get the values of

$D_n\left[\frac{\cos(\theta_i-\varphi_j)}{\cos(\theta_i+\varphi_j)}\right]$, $D_n\left[\frac{\sin(\theta_i-\varphi_j)}{\cos(\theta_i+\varphi_j)}\right]$ and $D_n\left[\frac{\cos(\theta_i-\varphi_j)}{\sin(\theta_i+\varphi)}\right]$ respectively.

THEOREM 4. $D_n[\tan(\theta_i+\varphi_j)]$

$$=(-1)^{n/2}\cos(\theta_1+\cdots+\theta_n+\varphi_1+\cdots+\varphi_n)\prod_{\substack{i,j=1\\(i>j)}}^{n}\sin(\theta_i-\theta_j)\sin(\varphi_i-\varphi_j)$$

$$\times\prod_{i,j=1}^{n}\sec(\theta_i+\varphi_j), \qquad \text{if } n \text{ is even}$$

$$=(-1)^{\frac{n-1}{2}}\sin(\theta_1+\cdots+\theta_n+\varphi_1+\cdots+\varphi_n)\prod_{\substack{i,j=1\\(i>j)}}^{n}\sin(\theta_i-\theta_j)\sin(\varphi_i-\varphi_j)$$

$$\times\prod_{i,j=1}^{n}\sec(\theta_i+\varphi_j), \qquad \text{if } n \text{ is odd.}$$

By making the substitutions exp. $(2i\theta_k) = x_k$, and exp. $(-2i\varphi_k) = y_k$ we see that $D_n[\tan(\theta_i + \varphi_j)] = \frac{1}{i^n} D_n\left[\frac{x_i - y_j}{x_i + y_j}\right]$,

and by using lemma 2 we can get the required result.

By changing θ_i to $\frac{1}{2}\pi + \theta_i$ we can evaluate $D_n[\cot(\theta_i + \varphi_j)]$.

3. Various other determinants can also be evaluated by suitable substitutions in lemmas (1) and (2). Thus, for instance, results similar to those in the above theorems can be obtained where instead of circular functions we have hyperbolic functions.

GLEANINGS

Another characteristic calls for mention here, as several writers and artists (some from Hollywood) have asked that it be treated—the sex life of the great mathematicians. In particular these enquirers wish to know how many of the great mathematicians have been perverts—a somewhat indelicate question, possibly, but legitimate enough to merit a serious answer in these times of preoccupation with such topics. None. Some lived celibate lives, usually on account of economic disabilities, but the majority were happily married and brought up their children in a civilized, intelligent manner. The children, it may be noted in passing, were often gifted far above the average. A few of the great mathematicians of bygone centuries kept mistresses when such was the fashionable custom of their times. The only mathematician discussed here whose life might offer something of interest to a Freudian is Pascal.

E. T. BELL in *Men of Mathematics*.

"What say you," pursued he (Socrates) "upon the Sun's return after winter to revisit us; and that as the fruits of one season wither and decay, he ripens new ones to succeed them? That having rendered man this service, he retires, lest he should incommode him by excess of heat; and then after having removed to a certain point, which he could not pass without putting us in danger of perishing with cold, that he returns in the same track to resume his place in those parts of the heavens where his presence is most beneficial to us? And because we could neither support the cold nor heat, if we were to pass in an instant from the one to the other, do you not admire, that whilst this star approaches and removes so slowly, the two extremities arrive by almost insensible degrees? Is it possible not to discover, in this disposition of the seasons of the year, a providence and goodness, not only attentive to our necessities, but even our delights and enjoyments?"

From ROLLIN'S *Universal History*

THE MAXIMUM TERM OF AN ENTIRE SERIES

BY

S. M. SHAH, *Muslim University, Aligarh.*

§1. Let $f(z)=\sum_{0}^{\infty} a_n z^n$ be an integral function ; $\mu(r)$ the maximum term for $|z|=r$ and $\nu(r)$ its rank. It is known[1] that if $f(z)$ be of finite order $\rho>0$

$$\liminf_{r\to\infty} \frac{\nu(r)}{\log \mu(r)} \leqslant \rho \leqslant \limsup_{r\to\infty} \frac{\nu(r)}{\log \mu(r)} \qquad \dots \quad (1)$$

We prove here a result which includes (1)

THEOREM 1. *If $f(z)$ be of order ρ, $0 \leqslant \rho \leqslant \infty$ then*[2]

$$\liminf_{r\to\infty} \frac{\nu(r)}{\log \mu(r)} \leqslant \lambda \leqslant \rho \leqslant \limsup_{r\to\infty} \frac{\nu(r)}{\log \mu(r)} \qquad \dots \quad (2)$$

where $\lambda = \liminf_{r\to\infty} \{\log \log M(r)/\log r.\}$

Corollary: *A necessary and sufficient condition that $f(z)$ be of order ρ and of regular growth is that* $\nu(r)/\log \mu(r) \to \rho$, as $r\to\infty$.

§2. We require a lemma.

Let (i) $X(x)$ *be a real function, positive and integrable* L *in any interval* (Δ, r) *where* $\Delta>0$;

(ii) $X(r) \geqslant 1$ *for all large* r ;

(iii) $\limsup_{r\to\infty} \frac{\log X(r)}{\log r} = A$; $\liminf_{r\to\infty} \frac{\log X(r)}{\log r} = B$;

(iv) $I(r) = \int_{\Delta}^{r} \frac{X(x)dx}{x}$;

then

$$\liminf_{r\to\infty} \frac{X(r)}{I(r)} \leqslant B \leqslant A \leqslant \limsup_{r\to\infty} \frac{X(r)}{I(r)} \qquad \dots \quad (3)$$

The proof is similar to that of the lemma of my paper[3] and is omitted.

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[1] G. Polya and G. Szego: Aufgaben und Lehrsatze aus der Analysis, II Ex. 59—60 (P 9).

[2] $M(r)$, $n(r)$, $N(r, a)$, $\overset{+}{\log} A$ have their usual meanings. See E. C. Titchmarsh Theory of Functions (1939) ch. 8.

[3] Note on a Theorem of Polya, *J. I. M. S.*, 5 (1941) 189—191.

§3. PROOF OF THEOREM 1.

We have
$$\log \mu(r) = A_1 + \int_\Delta^r \frac{\nu(t)}{t} dt$$

Putting $X(r) = \nu(r)$ in the lemma[1] and noting that[2]
$$\liminf_{r\to\infty} \frac{\log \nu(r)}{\log r} = \lambda ; \quad \limsup_{r\to\infty} \frac{\log \nu(r)}{\log r} = \rho,$$
we obtain
$$\liminf_{r\to\infty} \frac{\nu(r)}{\log \mu(r)} \leqslant \lambda \leqslant \rho \leqslant \limsup_{r\to\infty} \frac{\nu(r)}{\log \mu(r)}$$

§4. Let
$$\lambda_1 = \liminf_{r\to\infty} \frac{\overset{+}{\log}\, n(r)}{\log r} ; \quad \rho_1 = \limsup_{r\to\infty} \frac{\overset{+}{\log}\, n(r)}{\log r}$$

If $n(r) > 0$ for some value of r then $n(r) \geqslant 1$ for all large r and we have from the lemma
$$\liminf_{r\to\infty} \frac{n(r)}{N(r,0)} \leqslant \lambda_1 \leqslant \rho_1 \leqslant \limsup_{r\to\infty} \frac{n(r)}{N(r,0)} \quad \ldots \quad (4)$$

If $n(r) = 0$ for all values of r then $\lambda_1 = \rho_1 = 0$ and the extreme two terms of inequality (4) are also zero.

§5. We now suppose $f(z)$ is of finite and positive order ρ. Let
$$\alpha = \limsup_{r\to\infty} \left\{ \log M(r)/r^\rho \right\} ; \quad \beta = \liminf_{r\to\infty} \left\{ \log M(r)/r^\rho \right\}$$
$$\gamma = \limsup_{r\to\infty} \left\{ \nu(r)/r^\rho \right\} ; \quad \delta = \liminf_{r\to\infty} \left\{ \nu(r)/r^\rho \right\}$$
$$L = \limsup_{r\to\infty} \left\{ n(r)/r^\rho \right\} ; \quad l = \liminf_{r\to\infty} \left\{ n(r)/r^\rho \right\}$$

We prove some relations between these numbers.

THEOREM 2. *If $f(z)$ be of order ρ, $0 < \rho < \infty$*

(i) $\gamma \geqslant \alpha\rho \geqslant \beta\rho \geqslant \delta$

(ii) $\alpha \geqslant \dfrac{L \log a + l}{a\rho}$ *where*[3] *a is any number* $\geqslant 1$.

---

[1] We suppose that $f(z)$ does not reduce to a constant. Hence $\nu(r) \geqslant 1$ for all large r.

[2] J. M. Whittaker: The lower order of Integral Functions, *J. L. M. S.*, 8, (1933) 20—27.

[3] See also M. L. Cartwright *Proc. London Math. Soc.* 33. (1932) 209—224 (Theorems 4 and 9).

Proof. Let $a \geqslant 1$ and suppose $0 < \delta < \infty$.

$$\log M\left(ra^{1/\rho}\right) \geqslant A_3 + \int_{r_0}^{ra^{1/\rho}} \frac{\nu(x)}{x} dx$$

where $\nu(x)/x^\rho > \delta - \varepsilon$ for $x \geqslant r_0$.

Hence $\log M\,(ra^{1/\rho}) \geqslant A_3 + \frac{(\delta - \varepsilon)r^\rho}{\rho} + \frac{\nu(r)}{\rho} \log a.$

$$\therefore \quad \alpha \geqslant \frac{\gamma \log a + \delta}{\rho a}; \quad \beta \geqslant \frac{\delta \log a + \delta}{\rho a}$$

which can be shown to hold also when $\delta = 0$ or ∞.

Taking $a = 1$ we get $\beta \geqslant \delta/\rho$. Similarly we get

$$\alpha \geqslant \frac{L \log a + l}{\rho a}$$

To prove $\alpha \leqslant \gamma/\rho$ we may suppose $\gamma < \infty$.

$$\log M(r) \sim \log \mu(r) = A_4 + \int_{r_1}^{r} \frac{\nu(t)}{t} dt < A_5 + (\gamma + \varepsilon) \int_{r_2}^{r} t^{\rho - 1} dt$$

and hence the theorem is proved.

GLEANINGS

When all things were ready, the moment they (The Athenians) were going to set sail, wholly unsuspected by the enemy, who were far from surmising that they would leave Sicily so soon, the moon was suddenly eclipsed in the middle of the night, and lost all its splendour; which terrified Nicias and the whole army, who, from ignorance and superstition, were astonished at so sudden a change, the causes of which they did not know, and therefore dreaded the consequences of it. They then consulted the soothsayers; and who, being equally unacquainted with the reasons of this phenomenon, only augmented their consternation. It was the custom, after such accidents had happened, to suspend their enterprises but for three days. The soothsayers pronounced that he must not set sail till three times nine days had past; these are Thucydides' words, which doubtless was a mysterious number in the opinion of the people. Nicias, scrupulous to a fault, and full of a mistaken veneration for these blind interpreters of the will of the gods, declared that he would wait a whole revolution of the moon, and not return till the same day of the next month; as if he had not seen the planet very clearly the instant it had emerged from that part which was darkened by the interposition of the Earth's body.

From ROLLIN: *Universal History*

ON THE PROBABILITY OF OBTAINING k SETS OF CONSECUTIVE SUCCESSES IN n TRIALS

BY

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Let p be the probability of the occurrence of an event E in any one trial and q the probability of its failure so that $p+q=1$. The probability of obtaining a run of r or more consecutive successes (the occurrence of the event being counted a success) in n trials is given by * $(1-z_n)$ where z_n is the coefficient of x^n in

$$\frac{1-p^r x^r}{1-x+p^r q x^{r+1}} \; .$$

We shall generalise this result to find the probability of obtaining any number, k say, of sets of r or more consecutive successes in n trials.

Now, consider the ways in which we can get k sets of r or more consecutive successes in $n+1$ trials. This can happen in two mutually exclusive ways.

(i) We may get k sets in $n+1$ trials only.

(ii) We may get k sets in the first n trials and the $(n+1)^{th}$ trial may be a failure if the nth trial is the last of a set of $(r-1)$ consecutive successes, k sets of r consecutive successes having been completed earlier in the series of trials, it being immaterial in any other case whether the $(n+1)^{th}$ trial is a success or failure.

Let $y_{k,n}$ denote the probability of obtaining k sets of r (or more) consecutive successes in n trials. Then the probability of contingency (i) is $y_{k-1,\, n-r}\, qp^r$ because we must have $k-1$ sets in the first $(n-r)$ trials, then a failure and then r consecutive successes. The probability of contingency (ii) is the sum of the following probabilities:

(a) the probability of getting k sets in the first $(n-r)$ trials, then a failure followed by $(r-1)$ consecutive successes and failure at the $(n+1)^{th}$ trial is $y_{k,\, n-r} \cdot q^2 p^{r-1}$;

* J. V. Uspensky, *Introduction to Mathematical Probability* (McGraw Hill, 1937) page 79. See also Whitworth, *Choice and Chance.*

(*b*) the probability of getting k sets in n trials minus the probability of getting k sets in the first $(n-r)$ trials followed by a failure and $(r-1)$ consecutive successes in order $= y_{n,k} - y_{k,n-r} \cdot qp^{r-1}$.

The sum of the probabilities of contingencies (i) and (ii) must be the probability of getting k sets in $(n+1)$ trials. Thus we have the recurrence relation,

$$y_{k,n+1} = y_{k-1,n-r} \cdot qp^r + y_{k,n-r} \cdot q^2p^{r-1} + y_{k,n} - y_{k,n-r} \cdot qp^{r-1} \quad \ldots \quad (1)$$

This equation can be reduced to

$$\phi_k(x)\,[1 - x - q^2p^{r-1}x^{r+1} + qp^{r-1}x^{r+1}] - y_{k,r}x^r = qp^rx^{r+1}\phi_{k-1}(x)$$

where $\phi_k(x) = y_{k,0} + y_{k,1}x + y_{k,2}x^2 + \cdot\cdot\cdot\cdot\cdot\cdot$ ... (2)

From (2) we obtain

$$(1 - x + qp^rx^{r+1})^k\phi_k(x) = (qp^rx^{r+1})^k\ \phi_0(x)$$

$\phi_0(x)$ is the expression (due to Whitworth) given in the opening para of the paper and

$$\phi_k(x) = \frac{[qp^rx^{r+1}]^k(1-p^rx^r)}{[1-x+p^rqx^{r+1}]^{k+1}}$$

The required probability $y_{k,n}$ is thus the coefficient of x^n in $\phi_k(x)$ expanded in ascending powers of x.

GLEANINGS

What is reported of Euclid the Magarian, explains still better how high the passion of Socrates' disciples ran to receive the benefit of his instructions. There was at that time an open war between Athens and Magara, which was carried on with so much animosity, that the Athenians obliged their generals to take an oath to lay waste the territory of Magara twice a year, and prohibited the Magarians to set foot in Attica upon pain of death. This decree could not extinguish nor suspend the zeal of Euclid. "He left his city in the evening in the disguise of a woman, with a veil upon his head, and came to the house of Socrates in the night, where he continued till the approach of day, when he returned in the same manner he came.

From ROLLIN'S *Universal History*

A PROBLEM IN COMBINATIONS

BY

V. NARASIMHA MURTI, M.A., *Andhra University.*

§1. The problem is "Can m persons keep a watch on m days, n persons keeping a watch daily, so that any two persons go together on two and only two days."

Dr. Hansraj Gupta* gave solutions of this problem for $n=3, 4, 5$ and 6. Therein he has given, what are called "elegant solutions" for $n=3$, 4 and 5. For $n=6$ he has not obtained any "elegant solution". Here I show that there can be no elegant solution for $n=6$.

When one day's arrangement is taken, and if the other $(m-1)$ arrangements can be formed by adding $1, 2, \cdots, m-1$, to the members of this arrangement, the solution will be called an "elegant solution". We also know that for the existence of a Solution of the above problem, m must be equal to $\frac{n(n-1)}{2}+1$. We denote m persons by the numbers of $1, 2, 3, \cdots, m$.

§2. ***n*** $=$ ***3***, ***m*** $=$ ***4***. Let the first row of the "elegant solution" be $(1, 2, 2+x_1)$, without loss of generality. The three differences formed out of these three numbers are $1, x_1, x_1+1$. These must neither be equal, nor two of them equal and the third complementary to them. [Two numbers a and b are said to be complementary if $a+b\equiv 0 \pmod{m}$]. Hence for $x_1=1$ or 2 we get the following solution

1, 2, 3,
2, 3, 4,
3, 4, 1,
4, 1, 2.

§3. ***n*** $=$ ***4***, ***m*** $=$ ***7***. Let the first row be $(1, 2, 2+x_1, 2+x_1+x_2)$. The 6 differences formed out of these 4 numbers are $1, 1+x_1, 1+x_1+x_2, x_1, x_1+x_2$, and x_2; and not more than two of these differences can be either equal or complementary. Here $x_1+x_2\leqslant 5$, since we can arrange the numbers in ascending order of magnitude. The possible sets are (1) $x_1=1, x_2=2$; (2) $x_1=1, x_2=3$; (3) $x_1=2, x_2=3$;

* *The Mathematics Student* Vol. VIII, No. 3, p. 31.

(4) $x_1=3$, $x_2=2$. The sets (3) and (4) give the same solutions as the sets (1) and (2). They are

1, 2, 3, 5,	1, 2, 3, 6,
2, 3. 4, 6,	2, 3, 4, 7,
3, 4, 5, 7,	3, 4, 5, 1,
4, 5, 6, 1,	4, 5, 6, 2,
5, 6, 7, 2,	5, 6, 7, 3,
6, 7, 1, 3,	6, 7, 1, 4,
7, 1, 2, 4,	7, 1, 2, 5,

§4. ***n = 5, m = 11.*** Let the first row be (1, 2, $2+x_1$, $2+x_1+x_2$, $2+x_1+x_2+x_3$,) where $x_1+x_2+x_3 \leqslant 9$. Imposing the conditions for an elegant solution upon the 10 differences formed out of the above 5 numbers, the following possible sets are obtained:

(1) $x_1=1,\ x_2=2,\ x_3=3$;
(2) $x_1=1,\ x_2=4,\ x_3=3$;
(3) $x_1=2,\ x_2=3,\ x_3=4$;
(4) $x_1=4,\ x_2=3,\ x_3=2$.

The sets (3) and (4) give the same solutions as the sets (1) and (2). They are:

1,	2,	3,	5,	8,	1,	2,	3,	7,	10,
2,	3,	4,	6,	9,	2,	3,	4,	8,	11,
3,	4,	5,	7,	10,	3,	4,	5,	9,	1,
4,	5,	6,	8,	11,	4,	5,	6,	10,	2,
5,	6,	7,	9,	1,	5,	6,	7,	11,	3,
6,	7,	8,	10,	2,	6,	7,	8,	1,	4,
7,	8,	9,	11,	3,	7,	8,	9,	2,	5,
8,	9,	10,	1,	4,	8,	9,	10,	3,	6,
9,	10,	11,	2,	5,	9,	10,	11,	4,	7,
10,	11,	1,	3,	6,	10,	11,	1,	5,	8,
11,	1,	2,	4,	7,	11,	1,	2,	6,	9,

§5. ***n = 6, m = 16.*** Let the first row, as usual, be (1, 2, $2+x_1$, $2+x_1+x_2$, $2+x_1+x_2+x_2$, $2+x_1+x_2+x_3+x_4$,) where $x_1+x_2+x_3+x_4 \leqslant 14$. When the successive 16 differences are formed, it is found after calculation that there is no set of numbers x_1, x_2, x_3, x_4 which gives an elegant solution. Hence there is no elegant solution for $n=6$.

Similarly, by calculation, we may examine the existence of elegant solutions for $n=7$.

SOME ELLIPTIC FUNCTION FORMULAE

BY

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We have $\wp'(u+\omega_1)\cdot\wp'(u+\omega_2)$

$$= \left[-\frac{(e_1-e_2)(e_1-e_3)}{\{\wp(u)-e_1\}^2}\wp'(u)\right]\left[-\frac{(e_2-e_1)(e_2-e_3)}{\{\wp(u)-e_2\}^2}\wp'(u)\right]$$

$$= k\,(e_1-e_2)\{\wp(u)-e_3\}^2,$$

where $k = \dfrac{(e_1-e_2)(e_2-e_3)(e_3-e_1)}{[\{\wp(u)-e_1\}\{\wp(u)-e_2\}\{\wp(u)-e_3\}]^2}\cdot\wp'^2(u)$

$$\therefore\quad \sum\wp'(u+\omega_1)\,\wp'(u+\omega_2) = k\sum e_3^2(e_1-e_2)$$

$$= -\frac{\{(e_1-e_2)(e_2-e_3)(e_3-e_1)\}^2\,\wp'^2(u)}{[(\wp(u)-e_1)(\wp(u)-e_2)(\wp(u)-e_3)]^2}$$

$$= -16\,\frac{\{(e_1-e_2)(e_2-e_3)(e_3-e_1)\}^2}{\wp'^2(u)}$$

$$= -\frac{\wp'(u)\cdot\wp'(u+\omega_1)\cdot\wp'(u+\omega_2)\cdot\wp'(u+\omega_3)}{\wp'^2(u)}$$

Thus, we have

$$(1)\quad \frac{1}{\wp'(u)} + \frac{1}{\wp'(u+\omega_1)} + \frac{1}{\wp'(u+\omega_2)} + \frac{1}{\wp'(u+\omega_3)} = 0.$$

Also, by differentiation, this gives

$$(2)\quad \frac{\wp''(u)}{\wp'^2(u)} + \frac{\wp''(u+\omega_1)}{\wp'^2(u+\omega_1)} + \frac{\wp''(u+\omega_2)}{\wp'^2(u+\omega_2)} + \frac{\wp''(u+\omega_3)}{\wp'^2(u+\omega_3)} = 0.$$

We shall write for shortness,

$$\rho = \wp(u);\ \rho_1 = \wp(u+\omega_1);\ \rho_2 = \wp(u+\omega_2);\ \rho_3 = \wp(u+\omega_3)$$

Then, on differentiating (2) we have

$$\sum\frac{\wp'''}{\wp'^2} - 2\sum\frac{\wp''^2}{\wp'^3} = 0$$

where, $\sum\dfrac{\wp'''}{\wp'^2}$ means $\dfrac{\wp'''(u)}{\wp'^2(u)} + \dfrac{\wp'''(u+\omega_1)}{\wp'^2(u+\omega_1)} + \cdots + \dfrac{\wp'''(u+\omega_3)}{\wp'^2(u+\omega_3)}$

(i.e.) since $\wp''' = 12\,\wp\,\wp'$,

$$\sum\frac{\wp''^2}{\wp'^3} = 6\sum\frac{\wp}{\wp'} = 0.$$

$$(3)\quad \text{Thus}\quad \frac{\rho''^2}{\rho'^3} + \frac{\rho_1''^2}{\rho_1'^3} + \frac{\rho_2''^2}{\rho_2'^3} + \frac{\rho_3''^2}{\rho_3'^3} = 0$$

On differentiation, (3) gives

$$\sum \left(\frac{2 \wp'' \wp'''}{\wp'^3} - 3 \frac{\wp''^3}{\wp'^4} \right) = 0$$

(i.e.) $3 \sum \frac{\wp''^3}{\wp'^4} = 24 \sum \frac{\wp \wp''}{\wp'^2} = 24 \cdot 4 = 96,$

since, on differentiation of $\sum \frac{\wp}{\wp'} = 0$, we have $\sum \frac{\wp \wp''}{\wp'^2} = 4.$

(4) Thus, $\frac{\rho''^3}{\rho'^4} + \frac{\rho_1''^3}{\rho_1'^4} + \frac{\rho_2''^3}{\rho_2'^4} + \frac{\rho_3''^3}{\rho_3'^4} = 32$

On differentiating (4),

$$\sum \left(3 \frac{\wp''^2 \cdot 12 \wp \wp'}{\wp'^4} - 4 \frac{\wp''^4}{\wp'^5} \right) = 0$$

(i.e.) $\sum \frac{\wp''^4}{\wp'^5} = 9 \sum \frac{\wp \wp''^2}{\wp'^3} = 9 \sum \left\{ \frac{\wp}{\wp'} (4 \wp(2u) + 8 \wp) \right\}$

$$= 36 \cdot \wp(2u) \sum \frac{\wp}{\wp'} + 72 \sum \frac{\wp^2}{\wp'}$$

(5) $\therefore \sum \frac{\wp''^4}{\wp'^5} = 72 \sum \frac{\wp^2}{\wp'}$

Now, from the result $\sum \frac{\wp}{\wp'} = 0$, we have $4 = \sum \frac{\wp \wp''}{\wp'^2}$

Differentiating again, $\sum \frac{\wp''}{\wp'} = \sum 2 \frac{\wp \wp''^2}{\wp'^3}$

(i.e.) $\sum \frac{\wp \wp''^2}{\wp'^3} = 0$

(i e.) $0 = \sum \frac{\wp \wp''^2}{\wp'^3} = \sum \frac{\wp}{\wp'} (4\wp(2u) + 8\wp)$

$$= 8 \sum \frac{\wp^2}{\wp'} + 4\wp(2u) \sum \frac{\wp}{\wp'}$$

Thus $\sum \frac{\wp^2}{\wp'} = 0.$

(6) (i.e.) $\frac{\rho^2}{\rho'} + \frac{\rho_1^2}{\rho_1'} + \frac{\rho_2^2}{\rho_2'} + \frac{\rho_3^2}{\rho_3'} = 0$

Thus from (5), we have

(7) $\frac{\rho''^4}{\rho'^5} + \frac{\rho_1''^4}{\rho_1'^5} + \frac{\rho_2''^4}{\rho_2'^5} + \frac{\rho_3''^4}{\rho_3'^5} = 0$

Eliminating $\frac{1}{\rho'}, \frac{1}{\rho_1'}, \frac{1}{\rho_2'}, \frac{1}{\rho_3'}$ from (1), (2), (3), (7),

we have

$$(8) \qquad \begin{vmatrix} \frac{\rho''}{\rho'} & \left(\frac{\rho''}{\rho'}\right)^2 & \left(\frac{\rho''}{\rho'}\right)^4 & 1 \\ \frac{\rho_1''}{\rho_1'} & \left(\frac{\rho_1''}{\rho_1'}\right)^2 & \left(\frac{\rho_1''}{\rho_1'}\right)^4 & 1 \\ \frac{\rho_2''}{\rho_2'} & \left(\frac{\rho_2''}{\rho_2'}\right)^2 & \left(\frac{\rho_2''}{\rho_2'}\right)^4 & 1 \\ \frac{\rho_3''}{\rho_3'} & \left(\frac{\rho_3''}{\rho_3'}\right)^2 & \left(\frac{\rho_3''}{\rho_3'}\right)_4 & 1 \end{vmatrix} = 0.$$

Also, again, we can show by using the relations (1) (6) and $\sum \frac{\wp}{\wp'} = 0$, the result:

If α, β, γ, δ be ρ, ρ_1, ρ_2, ρ_3 taken in any order then, accents denoting differentiation w.r.t. the argument, we have,

$$\begin{aligned} -\alpha' & [\beta(\gamma^2-\delta^2)+\gamma(\delta^2-\beta^2)+\delta(\beta^2-\gamma^2)] \\ &= \beta'[\alpha(\gamma^2-\delta^2)-\alpha^2(\gamma-\delta)+(\gamma\delta^2-\gamma^2\delta)] \\ &= \gamma'[\alpha(\delta^2-\beta^2)-\alpha^2(\delta-\beta)+(\delta\beta^2-\delta^2\beta)] \\ &= \delta'[\alpha(\beta^2-\gamma^2)-\alpha^2(\beta-\gamma)+(\beta\gamma^2-\beta^2\gamma)] \end{aligned}$$

which can be rewritten

$$(9) \qquad \begin{aligned} -\alpha'(\beta-\gamma)(\gamma-\delta)(\delta-\beta) &= \beta'(\alpha-\gamma)(\gamma-\delta)(\delta-\alpha) \\ = \gamma'(\alpha-\beta)(\beta-\delta)(\delta-\alpha) &= \delta'(\alpha-\beta)(\beta-\gamma)(\gamma-\delta). \end{aligned}$$

If instead of using the equation (5) we use the relation $\sum \frac{\wp''}{\wp} = 0$ and solve as before, we get

$$(10) \qquad \begin{aligned} -\alpha' & [\beta(\gamma''-\delta'')+\gamma(\delta''-\beta'')+\delta(\beta''-\gamma'')] \\ &= \beta'[\alpha(\gamma''-\delta'')-\alpha''(\gamma-\delta)+(\gamma\delta''-\gamma''\delta)] \\ &= \gamma'[\alpha(\delta''-\beta'')-\alpha''(\delta-\beta)+(\delta\beta''-\delta''\beta)] \\ &= \delta'[\alpha(\beta''-\gamma'')-\alpha''(\beta-\gamma)+(\beta\gamma''-\beta''\gamma)] \end{aligned}$$

From the above results we can also easily obtain the following relations :

$$(11) \qquad \frac{\rho''^5}{\rho'^6} + \frac{\rho_1''^5}{\rho_1'^6} + \frac{\rho_2''^5}{\rho_2'^6} + \frac{\rho_3''^5}{\rho_3'^6} = 768\,\wp'(2u).$$

(12)
$$\begin{vmatrix} \rho & \rho^2 & \rho^4 & 1 \\ \rho_1 & \rho_1^2 & \rho_1^4 & 1 \\ \rho_2 & \rho_2^2 & \rho_2^4 & 1 \\ \rho_3 & \rho_3^2 & \rho_3^4 & 1 \end{vmatrix} = 4\rho(2u) \begin{vmatrix} \rho & \rho^2 & \rho^3 & 1 \\ \rho_1 & \rho_1^2 & \rho_1^3 & 1 \\ \rho_2 & \rho_2^2 & \rho_2^3 & 1 \\ \rho_3 & \rho_3^2 & \rho_3^3 & 1 \end{vmatrix}.$$

(13) If $x' = \wp'(u+\omega_1)$; $x'' = \wp''(u+\omega_1)$; similarly for y and z; and if we put
$$(\bar{x}, \underline{y}, z) = \begin{vmatrix} -x' & x'' & 1 \\ y' & -y'' & 1 \\ z' & z'' & 1 \end{vmatrix},$$

we can prove that:

$$(\bar{x}\ \underline{y}\ z) + (\bar{x}\ y\ \underline{z}) + (\underline{x}\ \bar{y}\ z) + (x\ \bar{y}\ \underline{z}) + (\underline{x}\ y\ \bar{z}) + (x\ \underline{y}\ \bar{z})$$
$$= 256\,\frac{[(e_1-e_2)(e_2-e_3)(e_3-e_1)]^2}{\wp'^3(u)}$$

While still at college Gauss had begun those researches in the higher arithmetic which were to make him immortal. His prodigious powers of calculation now came into play. Going directly to the numbers themselves he experimented with them, discovering by induction recondite general theorems whose proofs were to cost even him an effort. In this way he discovered the "gem of arithmetic," "*theorama aureum,*" which Euler also had come upon inductively, which is known as the law of quadratic reciprocity, and which he was to be the first to prove. (Legendre's attempted proof slurs over a crux.)

The whole investigation originated in a simple question which many beginners in arithmetic ask themselves: How many digits are there in the period of a repeating decimal? To get some light on the problem Gauss calculated the decimal representations of all the fractions $1/n$ for $n=1$ to 1000. He did not find the treasure he was seeking, but something infinitely greater—the law of quadratic reciprocity.

E. T. BELL in *Men of Mathematics.*

NOTES AND DISCUSSIONS.

Nurnberg Proof of Cauchy's General Principle of Convergence

In this note, I prove the sufficiency condition directly from a suitable Dedekind's section.

THEOREM: If, given $\varepsilon > 0$, we can find an n_0 such that

$$|\phi(n_2) - \phi(n_1)| < \varepsilon \text{ for all } n_2 > n_1 \geqslant n_0,$$

then the sequence $\phi(n)$ tends to a limit.

Taking $\varepsilon = 1$, we can find an m such that $|\phi(n) - \phi(m)| \leqslant 1$ for all $n \geqslant m$. Hence for all $n \geqslant m$, $\phi(m) - 1 \leqslant \phi(n) \leqslant \phi(m) + 1$.

So the sequence $\phi(n)$ is bounded; in other words, there are two numbers A and B such that $A < \phi(n) < B$ for all n. ... (1)

Divide the real numbers into two class L and R as follows:

If $a \geqslant \phi(n)$ only for a finite number of values of n, put a in the L class.

If $b \leqslant \phi(n)$ only for a finite number of values of n, put b in the R class.

Then if a number r escapes classification. there should be infinite number of values n_3 and n_4 for which $\phi(n_3) < r < \phi(n_4)$. ... (2)

If possible, let two numbers r and s escape classification, where $r > s$.

Then from (2), there are infinitely many values n_1 and n_2 for which $\phi(n_1) < s < r < \phi(n_2)$, and so $\phi(n_2) - \phi(n_1) > r - s$ for infinitely many values of n_1 and n_2. ... (3)

But this (3) is contrary to the condition given in the statement of the theorem.

Hence if at all, only one number will escape classification. Put this number (if it exists) in the L class. Now,

(a) Every number in L is less than every number in R;

(b) No real number escapes classification.

(c) From (1), A belongs to L and B belongs to R; so both the classes exist.

Hence this division is a section. Let α be the number corresponding to this section. Then, if $\varepsilon > 0$, $\alpha + \varepsilon$ belongs to R and $\alpha - \varepsilon$ to L. So, corresponding every $\varepsilon > 0$, there is an n_0, such that $\alpha - \varepsilon > \phi(n) < \alpha + \varepsilon$ for all $n \geqslant n_0$. Hence $\phi(n) \rightarrow \alpha$ as $n \rightarrow \infty$. The theorem is thus proved.

Calcutta University. S. SIVASANKARANARAYANA PILLAI.

A Note on Prof. Genese's Theorem and a simple proof

In 1891 Professor Genese of the National College, Aberstwyth-Wales, published the following theorem :—

With any point in the plane of an ellipse as centre two real cricles can be drawn in either of which triangles can be inscribed whose sides touch the ellipse; the radius R of either is given by $(R^2 - OS^2)(R^2 - OH^2) = 4. b^2. R^2$. (Q. 10879 Vol. LV of *Educational Times*).

In 1912 the same problem was again published by Mr. S. V. Ramamurti (Adviser to H. E. the Governer of Madras) in the following form:—(Q. 16782, Vol. XX. *E. T.* N. S.)

"The condition that an infinite number of triangles can be inscribed in a circle of radius R and centre S, so as to circumcribe a given ellipse of foci F, F′ and minor axis b is

$$(R^2 - SF^2)(R^2 - SF'^2) = 4. R^2. b^2."$$

But he was told that the theorem had already been propounded and solved. Divested of its conic-sectional garb the Theorem may be restated as follows:—If S and H are two isogonal conjugates w.r.t. a triangle ABC of which O is the cricumcentre $(R^2 - OS^2)(R^2 - OH^2) = 4. R^2. ll'$ where l and l are the perpendiculars from S and H on any one of the sides of the triangle. Four proofs have been published of this theorem :—(ET. Vol. LV and LVII).

(1) By Professor Greenstreet.

This proof uses trilinear co-ordinates and also trignometry and actually relies on Salmon's *Conic Sections* (Art 131. Ex. 2.)

(2) By Mr. V. Ramaswamy Iyer who uses Analytical Geometry applying the formula for the tangent to a circle from a point and using determinants.

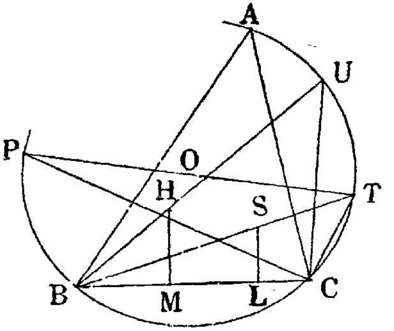
(3) Mr. Heppel

(4) Mr. Gallatly in *The Modern Geometry of the triangle*

} These use trigonometrical methods.

The following proof is submitted as a simple proof of the Theorem without requiring more knowledge than that possessed by a School Final student.

Produce BS and BH to meet the circumcircle in T and U. Join TC and UC. Join TO to meet the circumcircle in P. Join PC. Draw SL and HM $\perp$ to BC.



Then, Δ^s TPC and SBL are similar ($\because \angle$TPC = SBL and TCP = rt. angle = SLB.)

$\therefore$ 2 R. SL = SB. TC.

Similarly, 2 R. HM = HB. UC.

$\therefore$ 4 $R^2 ll'$ = SB. HB. TC. UC.

Again, $\angle$TSC = SBC + SCB.

= HBA + HCA. ($\because$ S and H are isogonal)

= BHC − A = BHC − BUC = UCH.

And $\angle$STC = $\angle$CUH.

$\therefore$ Δ^s TSC and UCH are similar.

$\therefore$ TC. UC = TS. UH

$\therefore$ 4 $R^2 l. l'$ = BS. BH. ST. HU.

= BS. ST. BH. HU.

= $(R^2 - OS^2)(R^2 - OH^2)$

Madras SIR V. RAMESAM

Note on Weierstrass's Theorem.

Mr. E. G. Phillips in his *Course of Analysis* gives the following proof of Weierstrass's Theorem, on page 32, §2·31.

THE THEOREM: Every bounded infinite aggregate has at least one limit point.

PROOF: To fix our ideas, consider only a linear infinite set of points included in the interval (a, b). Take any point c, such that $a < c < b$; then since (a, b) contains an infinite set, there must be an infinite set either in (a, c) or in (c, b) or in both. Suppose for definiteness, that there is an infinite set in (a, c). The set of points x in (a, c) is bounded above by c, so that c, and every number greater than c, is a rough upper bound. The aggregate of rough upper bounds determines a unique number ξ, the upper bound of the set. The number ξ has the properties,

$$\xi \geqslant x$$
$$\xi - \varepsilon < \text{at least one } x;$$

hence in the interval $(\xi - \varepsilon, \xi + \varepsilon)$ there is at least one point of the infinite set in (a, c) other than ξ. The number ξ is therefore a limit point.

Now let us consider the aggregate

$$0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots, \frac{1}{n} \ldots$$

This is an infinite aggregate bounded below by 0, and above by 1. Let c of the theorem be $\frac{1}{2}$, $a = 0$. Then (a, c) contains an infinite aggregate which is bounded by $\frac{1}{2}$ and 0. Therefore, arguing as in the above proof, $\frac{1}{2}$ and every number greater than $\frac{1}{2}$ from the aggregate of rough upper bounds which determines a $\xi = \frac{1}{2}$. But $\xi = \frac{1}{2}$ does not satisfy the condition of an upper bound or limit points. This proof of the theorem, therefore, is not strictly accurate and it fails when c is taken as a point of the aggregate.

Madras VEPA P. SARATHI

A Magic Square of Triangular Numbers.

The following is a magic square formed with the first 64 triangular numbers. The nth triangular number is given by $0+1+2+\cdots+n=\frac{1}{2}n(n+1)$, $n=0, 1, 2, \cdots$. The magic sum, viz, the sum of a row, column or diagonal is equal to 5460 which is itself a triangular number corresponding is $n=104$.

1378	171	496	1653	630	15	36	1081
105	820	703	6	325	2016	1275	210
861	120	3	666	1953	300	231	1326
190	1431	1596	465	10	595	1128	45
0	741	946	91	276	1225	1830	351
1711	406	153	1540	1035	66	21	528
435	1770	1485	136	55	990	561	28
780	1	78	903	1176	253	378	1891

Nurnberg. ALFRED MOESSNER

GLEANINGS

Statisticians, who have gone carefully into figures inform us that if young men who have just received a negative answer to a proposal of marriage (and with these must of course, be grouped those whose engagements have been broken off) 6·08 per cent clench their hands and stare silently before them, 12·02 take the next train to the Rocky Mountains and shoot grizzlies, while 11·07 sit down at their desks and become modern novelists. The first impulse of the remainder—and these, it will be seen constitute a great majority—is to nip off round the corner and get a good stiff drink.

From P. G. WODEHOUSE: *Hot Water.*

To make the idea of this reduction to an absurdity more vivid, the following historical anecdote is told:

A certain eastern king found one of his subjects intoxicated, and worshipping false gods. We are giving the king's reasoning and decision in the matter. Said the king:

1. This man is drunk, therefore he tells the truth ;
2. This man is worshipping false gods, therefore he lies ;
3. This man is telling the truth and lying at the same time.
4. This man's existence involves an absurdity.
5. Therefore, let this man be executed.

There are many mathematical principles whose demonstration reaches the absurdity point more slowly. If something assumed were to crawl slowly to an absurdity, and reach it after a hundred years, the thing assumed would be proved false. When the egg is hatched out, we discover whether it contains a bird or a snake.

From HAYN: *A Geometry Reader.*

BOOKS RECEIVED FOR REVIEW

1. Scripta Mathematica Library No. 5 *Galois Lectures*, New York, 1941.

2. LEVI, F. W: *Fundamentals of Analysis*, 1942, Calcutta University.

3. ————: *Finite Geometrical Systems*, 1942, Calcutta University.

4. ————: *Algebra*, Vol. I., 1942, Calcutta University.

5. M. K. NARASIMHAMURTI: *A Manual of Geometry*, 1942, Prabhakar Book Depot, Bangalore.

6. CREW, H: *Portfolio of Famous Physicists*, Scripta Mathematica, New York.

7. SURYANARAYANA IYER and SANTIAGO: *Projective Geometry.*

8. RAMESAM, V. and SARATHI, V. P: *A Supplement to Elementary Geometry*, 1942, K. Mahadevan, Madras.

9. SEGRE, B: *The Non Singular Cubic Surfaces*, 1942, Humphrey Milford, Oxford, 15/-

10. SESHADRI, T. N: *Handbook of Physical constants and Mathematical tables*, 1942, S. Viswanathan, Madras.

11. MAHAJANI, G. S: *Lessons in Elementary Analysis*, Third Edition, 1942, Aryabhushan Press, Poona.

12. S. R. GUPTA: *Elementary Analytical Statics*, S. Chand & Co., Delhi, Lahore, 1942.

13. M. V. JAMBUNATHAN: *Elementary Calculus*, Seshadri & Co., Bangalore, 1942, price Re. 1—4—0.

14. ————: *Elementary Analytical Geometry*, Seshadri & Co., Bangalore, 1941, Price Annas Twelve.

Many have imagined that there is one long chain from "The whole is greater than any of its parts," to calculating the ratio between the circumference of a circle and its diameter, and that every principle of the subject is a link in this single long chain. This is false in many ways. Nor will it do to represent the subject by a single tall tree with many branches. A better picture would be a grove of about 30 trees; and even this is incomplete until we admit birds and bees into our grove.

From HAYN: *A Geometry Reader.*

REVIEWS

Two Dimensional Potential Problems connected with Rectilinear Boundaries

BY DR. B. R. SETH, Lucknow University Studies No. XIII, 1939.

A number of problems of mathematical physics: two-dimensional motions of fluids, viscous or not, the torsions of a cylinder, St. Venant's flexure problem, some problems on membranes, etc., depend on the solution of the Problem of Neumann. In two dimensions, this consists in finding a solution of the Laplace's equation which satisfies the boundary condition $\frac{d\phi}{dn} = f$, $\frac{d}{dn}$ being the normal derivative on the boundary and f a function of x, and y depending on polynomials of the second degree at most. Now in applied mathematics it is not enough to know the existence of a solution, some construction of it is needed. In the problems treated by the author, the boundaries are rectilinear and it is possible to find a general method of constructing the solution. This method and its applications to important problems are treated in the book under review.

The method which is proposed is much more general than the method of images of Lord Kelvin, and not so laborious as the method suggested by Sokolnikoff, which is a direct application of the general theory of Potentials where the solution is represented by an integral. It is based on the works of Mitchell, Love, S. C. Kluyver, Leathem, Kotter, E. Trefftz, Bickley and Hodgkinson. Prof. B. R. Seth, to whom valuable contributions on the subject are due, gives a fairly good account of this new method, although I should have preferred him to insist on certain aspects of it a little more.

In essence it consists in finding out the form-finite form of the analytic function whose real part is the required solution. This solution contains a finite number of constants which can be calculated by an elegant conformal representation of the domain on another domain whose boundaries are also rectilinear.

A number of applications are treated in detail. The famous St. Venant's flexure problem is dealt with at great length. Very interesting examples of domains having one or more re-entrant angles are also considered and the student of hydrodynamics will find there many questions about which further research work seems very desirable.

C. RACINE

Handbook of Physical Constants and Mathematical Tables

BY T. N. SESHADRI, Presidency College, Madras. Published by S. VISWANATHAN, 2/10 Post Office Street, Madras G. T. pp. 48—iv.

This handbook is planned on the lines of the well-known Clark's tables. It is intended to furnish data, specially useful for students working in an Indian Laboratory. The long and continued use of Clark's tables would have naturally suggested to any one that an alternative handbook would command very little success. Mr. Seshadri's long experience as a teacher of science has made it possible for him to carry out the apparently impossible task of producing this neatly executed and successful handbook. Its forty eight pages are full of useful information, neatly arranged for quick and easy reference.

Opinions may easily differ regarding the material to be included and the order of presentation. It should however be admitted that the author has done his best to meet the demands of several subjects and different points of view.

A few misprints (*e.g.* on p. 30, $e/m = 5{\cdot}305 \times 10^{-17}$ e.s.u. per gm. *instead of* $5{\cdot}305 \times 10^{17}$ e.s.u. per gm.) should be corrected in the next edition. In the matter of spelling of scientific terms, it would be an advantage to follow some standard publication, like the *Proceedings of the Royal Society*.

The handbook is neatly got up, although one misses in its pages, the elegant printing of Clark's tables. But the publisher has spared no pains to make the handbook neat and attractive, in spite of great handicaps now prevalent in the printing industry.

The author and the publisher are to be congratulated on the high standard of this publication.

S. RAMACHANDRA RAO.

A Supplementary to Elementary Geometry

BY Sir V. RAMESAM, Kt, B.A, B.L., Retired Judge, Madras High Court and V. P. SARATHI, M.A., B.L., Pp. 114. 1942. (Published by K. Mahadevan, Madras) Rs. 2.

This little book furnishes striking evidence of the grip that Geometry can have over one's imagination. Sir Vepa Ramesam, a brilliant student of mathematics, had to submerse all his Mathematics except a little golden streak of Elementary Geometry in his active life as a lawyer and judge, but has now found time to give to the world the benefit of his geometrical assimilations. The reader should be thankful to the authors for the rare and beautiful theorems which are now presented for the first time in a form accessible even to undergraduates. The book also caters to the less ambitious student who is content with theorems that pay in examinations e.g., polarity, coaxal circles, maxima

and minima, properties of triangles, concurrency and collinearity. There is a proper emphasis on the basic theorems in geometry which are given in the early pages of the four books into which the booklet is divided.

The sources of the authors' inspiration are Casey's 'Sequel to Euclid' and the 'Educational Times Reprints', both now inaccessible to the ordinary reader. The Caseyan stunt of introducing Ratio and Proportion, and Circles as early as Books I and II dealing with theorems on straight lines, is adopted with approval and, in the reviewer's opinion, sometimes overdone.

Feuerbach's Theorem and the orthopole are the favourite themes of Sir Vepa who † claims to have collected more than sixty geometrical proofs of Feuerbach's theorem. He is particularly fascinated by Youngman's theorems mentioned in Bk. I, 52 and his own deduction of Feuerbach's theorem (Bk. III, 9) from it. which he claims to be new, (p 56), and asserts as the simplest geometrical proof (*The Math. Student* Vol., IX, p. 130). In Bk. III, 37, it is proved that the nine point circles of AIB, BIC, CIA pass through the in-Feuerbach point, (I being the incentre) much in the same way as Beard's proof as presented in a condensed and modified form in *The Math. Student* Vol. IX pp. 18, 19, where it is utilised for proving Feuerbach's theorem by the addition of one more obvious step. The authors have not chosen to take this step and furnish another proof of Feuerbach's theorem which is bound to impress any reader as much simpler, taken all in all, than the deduction of Feuerbach's theorem from Youngman's theorem.

The existence of the 'orthopole' is proved in Bk. I, 32, while further properties are discussed in Bk. III 24, 41, 45, 47, 48 and Bk. IV 21. Cor. 2. With the help of Pascal's theorem, an interesting result is proved, viz., the orthopole of a circumdiameter of A B C with respect to the triangle lies on the "reciprocal" to this diameter. If the equation of a line in areal co-ordinates be $lx+my+nz=o$, the line $x/l+y/m+z/n=o$ is called by Goormaghtigh the "reciprocal" to the first line. These lines are referred to as 'Seitengegengeraden' in the *Encyclopadie der Mathematischen Wissenschaften:* Neuere Driecks Geometrie (see also *J. I. M. S.* Vol. XVIII Part II pp. 106-109). The theorem proved in Bk. IV., 21, Cor. 2 is a particular case of Goormaghtigh's theorem:

The reciprocal line of the image in the circumcentre, of any line *m* passes thro' orthopole *M* of *m*.

Difficult and fundamental theorems on isogonal conjugates due to Genese and Ramaswami Iyer are proved in Bk. III, 49 (the reference to this in the index is an error, the page reference ought to be 85 instead of 86) and 56, involving an abundant use of rectangle properties of circles and similar triangles. There is no reason why they should be placed here, for they may as well occur in Bk. II. quite consistently with the Caseyan scheme which accommodates rectangle properties of circles and similar triangles long before Ratio and proportion and Circles.

† The Mathematical Gazette, Vol. XXV, p. 225.

The book concludes with Pascal's theorem and Desargues' theorem proved by means of Menelaus' theorem, which, however, does not find a place in the booklet. The omission of Ceva's and Menelaus' theorems is unfortunate. Such important theorems deserve attention and elaboration in a book of this kind. The enunciation of Pascal's theorem is indifferent. On the other hand there is a good introduction to Desargues' theorem as a plane section of a space configuration of lines and planes through five points, originally due to Von Staudt. It is to be wondered how many teachers attempt it in the class room.

Regarding elementary theorems the authors have tried to improve on existing proofs, with a good measure of success, witness for example, Bk. II 33. But there is scope for improvement in some other cases.

The proof of the converse theorems in Book II, 14 viz., 'if AC, BD intersect in E such that $AE\cdot EC = BE\cdot ED$, then, A, B, C, D lie on a circle', may be improved thus:

If the perpendicular bisectors of AC, BD meet in S and SE be joined, then $SB^2 - SE^2 = BE\cdot ED = AE\cdot EC = SA^2 - SE^2$, and so $SB = SA = SC = SD$.

This proof is given in the *New Geometry for High Schools* (pp. 517, 518), published in 1928 by Srinivasa Varadachari & Co., Madras.

In Book III, 14 are discussed some loci with precise demarcations. They were first pointed out more than twentyfive years ago in *The Mathematical Gazette* and since then it became a fashion to incorporate them in some text-books. But the precision suddenly stops when it comes to the locus of the 9-points centre, given the base and the vertical angle. Even here, the locus is not a complete circle, being terminated by the mid-points of the joins of the circumcentre to the extremities of the orthocentre-locus.

Again, for the converses of the theorems in Book I 42, 43, 55, direct proofs are available and deserve adoption. According to J. E. Blamey's statements, Herr Lietzmann in *Altes und Neues vom Kreis* (Teubner) has given an excellent proof of the converse proposition of the cyclic quadrilateral and Blamey himself has worked out on Lietzmann's model a proof of the theorem: Given $A\hat{C}B = A\hat{D}B$, and C, D on the same side of AB, then, ABCD is cyclic (Math. Gazette Vol. XXI, p. 231). Here is an illustration. The parallelism in proofs of two dual theorems Book I, 43, 55 is very striking and may be exhibited thus:

If $\hat{A}+\hat{C}=\hat{B}+\hat{D}$ in a quadrilateral ABCD, then a circle can be circumscribed about it.	If $AB+CD=AD+BC$ in a quadrilateral, then a circle can be inscribed in it.
Proof:—Let $A>B$; then $A-B = D-C$, and $D>C$. At B make $A\hat{B}O = D\hat{A}B$, and at C make $D\hat{C}O = A\hat{D}C$.	Proof:—Let $AB>BC$; then $AB-BC = AD-CD$, and $AD>CD$. On BC produced take E so that $BE=BA$; on DC produced take F so that $DF=DA$.

Then OB=OC, and the perpendicular bisectors of AB, BC, CD bisect also the angles of the triangle formed by OB, OC and AD, and meet at a point equidistant from A, B, C, and D.

Then CE=CF and the bisectors of the angles ABC, BCD, CDA are the perpendicular bisectors of the sides of AEF and so meet at a point equidistant from AB, BC, CD, and DA.

On the whole, one can have nothing but admiration for the authors to whom this work must have been a labour of love, and who have demonstrated the possible heights one can reach with such limited equipment as rectangle properties of circles and similar triangles. Some of the rarest gems in the literature of the subject, it has been found possible to include in this work of less than six score pages expounding nearly two hundred theorems great and small. A careful study of the work is sure to inspire the reader to do a bit of geometrical mountaineering himself and attempt to visit the immortal peaks where Pascal, Steiner, and Hamilton hold their sway.

Mysore. A. A. KRISHNASWAMI AYYANGAR

Portraits of Famous Physicists with biographical accounts,

BY HENRY CREW, Scripta Mathematica, New York, 1942, Price $ 375.

The Mathematical world already owes a debt of gratitude to the organizers of the *Scripta Mathematica* for a series of "Portraits of eminent mathematicians," each accompanied by a brief biographical sketch. The present portfolio of a dozen of the world's great physicists is issued as the fourth of the series on "pictorial mathematics" and maintains the same high standards that one has learnt to associate with the earlier series. The pictures are, as usual, on art paper while the biographical details are on a separate folder illustrated, in many cases, with sketches relating to the life of the scientist or a facsimile reproduction of his writings.

The series begins with three versatile men who achieved greatness in several fields, notably in Astronomy and Physics—Galileo, Christiaan Huygens and Sir Isaac Newton. Galileo with a telescope made by himself discovered spots on the sun, 4 of Jupiter's satellites, and noted the "triple nature of Saturn" which remained a puzzle for 50 years till Huygens resolved the appendages into the rings of Saturn. Galileo died on 8—1—1642, three days after the birth of Newton and yet he had seized on momentum as the fundamental quantity in mechanics and enunciated the first two laws of motion. Huygens will always be associated with the wave front and optical phenomena like double refraction, but his great contribution to Science was the pendulum clock invented at the age of 28. The portrait of Newton is the one by Vanderbank in the National Museum gallery, London and shows him in a thoughtful mood seated in front of a open book and a star globe.

England, which heads the list of portraits, makes two more contributions in the persons of Michael Faraday who unaided by academic training discovered the fundamental principles upon which the dynamo and the telephone depend and upon which Maxwell erected the electromagnetic theory of light and guided Hertz to the discovery of wireless telegraphy, and J. P. Joule who just a hundred years ago (1843) measured the mechanical equivalent of heat. The frugal Scotch make but one contribution, but he is the immortal James Clerk Maxwell after whom the unit of magnetic flux was named in 1932 as a mark of international appreciation. The Germans are represented by H. R. Hertz and by R. J. E. Clausius the inventor of the concept of entropy in thermodynamics. We meet two French Scientists-Andre Marie Ampere whose rule regarding the magnetic field due to a current element must be well known to students of Physics, and Augustin Fresnel who died in 1827 at the age of 39 after having established the wave theory of light on an experimental basis.

Of those who hail from the land of the *Scripta* we have but two, Josiah Willard Gibbs who made valuable contributions to thermodynamics but comes close to the student world through his classical work on "Vector Analysis" revised by one of his ablest students E. B. Wilson, and Henry Augustus Rowland whose grating has made spectroscopy what it is to-day.

These portraits were planned in 1937 and took account only of outstanding physicists not then living which explains the absence of Sir J. J. Thomson and Lord Rutherford. I have not been able to discover the order of arrangement of the portraits, but if arranged in the order of the dates of birth, there is only one inversion at the very end.

A. N. RAO.

Lessons in Elementary Analysis

Third Edition by G. S. MAHAJANI, Ph. D. (CANTAB), Principal, Fergusson College, Poona. 1942, pp. 294 Price Rs. 6-4-0.

It is a regrettable fact that there is no suitable text-book by Indian authors covering the syllabus in Analysis for the Honours degree courses of Indian Universities. The book under review rectifies this defect to some extent. The author points out in his preface to the first edition that the book was written mainly for the pass course students. But the book as it stands is more suited to the Honours than to pass students even though it does not cover completely the whole course for Honours students. The presentation of the topics covered by the book is both lucid and precise.

A few details deserve mention. In the definition of the limit of a function $f(x)$ at a point $x=a$ (p. 35), the author does not explicitly omit the value of $f(x)$ at $x=a$. He points out that this is to be the case in his later remarks. But a reader might be led to think after a perusal of the whole matter,

that the point raised above is not an essential one, and the author points out the fact somewhat casually. On the other hand, no amount of emphasis will be too much in drawing the attention of students to the fact that the notion of a *limit at a point* is *entirely independent* of the *value of the function at that point.* This being so, it is desirable that the above point be explicitly mentioned in the definition of the limit. Secondly, the chapter on limits would be more complete if the author had added a few elementary notions regarding point sets, limit points and their relation to upper and lower limits and a statement and proof of the conditions for the existence of the limit of a function at a point.

The notion of an infinitesimal as something which vanishes ultimately appears to be superfluous once the idea of a limit has been made precise. The orders of infinitesimals are merely measures of the rapidity of approach to a limit and belong properly to the domain of ideas like the orders of infinity. All the results in chapter III could be defined or proved with the notion of a limit alone. The so called indeterminate forms also come under the notion of limits. Lastly, the chapter on Riemann Integration would gain by a statement and proof of Darboux's theorem.

As already pointed out, the book, is well-written; the printing is very neat; and it satisfies, though partially, a really felt need of the Honours students of Indian Universities for a good text-book on Analysis.

V. G.

GLEANINGS

Probably the most striking case in history is that of the Bernoulli family, which in three generations produced eight mathematicians, several of them outstanding, who in turn produced a swarm of descendants about half of whom were gifted above the average and nearly all of whom, down to the present day, have been superior human beings. No fewer than 120 of the descendants of the mathematical Bernoullis have been traced genealogically, and of this considerable posterity the majority achieved distinction—sometimes amounting to eminence in the Law, Scholarship, Science, Literature, the Learned Professions, administration, and the arts. None were failures. The most significant thing about a majority of the mathematical members of this family in the second and third generations is that they did not deliberately choose mathematics as a profession but drifted into it in spite of themselves as a dipsomaniac returns to alcohol.

From BELL, *Men of Mathematics.*

QUESTIONS FOR SOLUTION

1808. (B. SEETHARAMA SASTRY): Find a function $\phi(x,y)$ such that the plane transformation T defined by

$$\xi = x^p y^q, \ \eta = \phi(x,y) \ (p,q > 0)$$

may be a cyclic transformation of the third order.

1809. (SAHIB RAM MANDAN): Show that the vertices of the 8 cones belonging to the system of quadrics $\lambda^2 S_1 + \lambda S_2 + S_3 = 0$ where S_1, S_2, S_3 are the point-equations of any 3 quadrics, form an associated set of 8 points.

1810. (SAHIB RAM MANDAN): A variable conic of a confocal system meets a given line in P,Q. Prove that the normals at P,Q to the conic meet on a fixed line. Identify this line.

1811. (A. O. AUGUSTINE): Let S_1, S_2, S_3 the three given conicoids, and A_{ij}, B_{ij}, C_{ij} and D_{ij} the vertices of the common self-conjugate tetrahedron of S_i and S_j. Prove that a unique conicoid exists which passes through the 9 points

$$A_{23}, B_{23}, C_{23}, D_{23}, A_{13}, B_{13}, C_{13}, A_{12}, B_{12}.$$

1812. (NATHAN ALTSHILLER-COURT): University of Oklahoma, U. S. A.

The lines AO, BO, CO, DO joining the vertices of the tetrahedron (T) = ABCD to the point O (not in a face of the tetrahedron) meet the respective faces of (T) in the points K, L, M, N, and the planes LMN, MNK, NKL, KLM in the points P, Q, R, S. Let Q_a, R_a, S_a be the traces of the lines AQ, AR, AS in the plane BCD; R_b, S_b, P_b the traces of the lines BR, BS, BP in the plane CDA, etc. (i) The four triads of planes BCD, LMN, $P_bP_cP_d$; CDA, MNK, $Q_cQ_dQ_a$; DAB, NKL, $R_dR_aR_b$; ABC, KLM, $S_aS_bS_c$ are coaxal; (ii) The four axes of these four triads are coplanar.

[*Note.* A similar problem in the plane was discussed in the Educational times, Reprints, vol. LXX (1899), p. 112, Q. 13930.]